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Some Results on Totally Real Maximal Spacelike Submanifolds of an Indefinite Complex Space Form

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Abstract

The purpose of this paper is to study curvature pinching of an n-dimensional compact totally real maximal spacelike submanifold M immersed in an indefinite complex space form $\bar{M}_p^{n+p}(c)$. We have shown that M is totally geodesic if the Ricci curvature R is less than or equal to $\frac{c}{4}(n-1)(1-n-2p), \quad n>1, \quad c \geq 0$. Moreover, if the scalar curvature $\rho \leq \frac{p(1-n)}{2}c, \quad n>1, \quad c \geq 0$.

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1 Introduction

A submanifold of a Kaehler manifold is called totally real (resp. holomorphic) if each tangent space of the submanifold is mapped into the normal space (resp. itself) by the almost complex structure of the Kaehler manifold [1]. A Kaehler manifold of constant holomorphic sectional curvature is called a complex space form.Let M be an n-dimensional totally real maximal spacelike submanifold isometrically immersed in a 2(n+p)-dimensional indefinite complex space form $\bar{M}_p^{n+p}(c)$ of holomorphic sectional curvature c and index 2p. We call M a spacelike submanifold if the induced metric on M from that of the ambient space is positive definite. Let J be the almost complex structure of $\bar{M}_p^{n+p}(c)$. An n-dimensional Riemannian manifold M isometrically immersed in $\bar{M}_p^{n+p}(c)$ is called totally real submanifold of $\bar{M}_p^{n+p}(c)$ if each tangent space of M is mapped into the normal space by the almost complex structure J. Let h be the second fundamental form of M in $\bar{M}_p^{n+p}(c)$ and let S denote the square of the length of the second fundamental form h. As far as the geometry of submanifolds is concerned, the second fundamental form plays an important role in determining some properties of the submanifold. The purpose of this paper is to study the geometry of an n-dimensional compact totally real maximal spacelike submanifold M immersed in an indefinite complex space form $\bar{M}_p^{n+p}(c)$. Our main result is:

Theorem 1.1. *Let M be an n-dimensional compact totally real maximal spacelike submanifold of* $\bar{M}_p^{n+p}(c)$ *. Then M is totally geodesic if the Ricci curvature R is less than or equal to* $\frac{c}{4}(n-1)(1-n-2p), \quad n > 1, \quad c \geq 0$*. Moreover, if the scalar curvature* $\rho \leq \frac{p(1-n)}{2}c, \quad n > 1, \quad c \geq 0$*, then M is totally geodesic.*

2 Local formulas

We choose a local field of orthonormal frames $\{e_1, \ldots, e_n; e_{n+1}, \ldots, e_{n+p}; e_{1*} = Je_1, \ldots, e_{n*} = Je_n; e_{(n+1)*} = Je_{n+1}, \ldots, e_{(n+p)*} = Je_{n+p}\}$ on $\bar{M}_p^{n+p}(c)$ in such a way that restricted to M, the vectors $\{e_1, \ldots, e_n; Je_1, \ldots, Je_n\}$ are tangent to M and the rest are normal to M. With respect to this frame field of $\bar{M}_p^{n+p}(c)$, let $\omega^1, \ldots, \omega^n; \omega^{n+1}, \ldots, \omega^{n+p}; \omega^{1*}, \ldots, \omega^{n*}; \omega^{(n+1)*}, \ldots, \omega^{(n+p)*}$ be the field of dual

frames. Unless otherwise stated, we shall make use of the following convention on the ranges of indices:$1 \leq A, B, C, D \leq n+p$; $\quad 1 \leq i, j, k, l, m \leq n$; $\quad n+1 \leq \alpha, \beta, \gamma \leq n+p$; and when a letter appears in any term as a subscript or a superscript, it is understood that this letter is summed over its range. Besides
$\varepsilon_i = g(e_i, e_i) = g(Je_i, Je_i) = 1$, when $1 \leq i \leq n$
$\varepsilon_\alpha = g(e_\alpha, e_\alpha) = g(Je_\alpha, Je_\alpha) = -1$, when $n+1 \leq \alpha \leq n+p$
Then the structure equations of $\bar{M}^{n+p}_p(c)$ are;
$d\omega^A + \sum_B \varepsilon_B \omega^A_B \wedge \omega^B = 0, \qquad \omega^A_B + \omega^B_A = 0, \qquad \omega^i_j = \omega^{i*}_{j*}, \qquad \omega^{i*}_j = \omega^{j*}_i$
$d\omega^A_B + \sum_C \varepsilon_C \omega^A_C \wedge \omega^C_B = \frac{1}{2}\varepsilon_C \varepsilon_D \bar{R}_{ABCD}\omega^C \wedge \omega^D$,
$\bar{R}_{ABCD} = \frac{c}{4}\varepsilon_C \varepsilon_D(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + J_{AC}J_{BD} - J_{AD}J_{BC} + 2J_{AB}J_{CD})$
where $\bar{R}_{ABCD}$ denote the components of the curvature tensor $\bar{R}$ on $\bar{M}^{n+p}_p(c)$. Restricting these forms to M we have;

$$\begin{aligned}
&\omega^\alpha = 0, \quad \omega^\alpha_i = \sum_i h^\alpha_{ij}\omega^i, \quad h^\alpha_{ij} = h^\alpha_{ji}, \quad d\omega^i = -\sum_j \omega^i_j \wedge \omega^j, \\
&\omega^i_j + \omega^j_i = 0, \quad d\omega^i_j = -\sum_k \omega^i_k \wedge \omega^k_j + \frac{1}{2}\sum_{kl} R_{ijkl}\omega^k \wedge \omega^l, \\
&R_{ijkl} = \bar{R}_{ijkl} - \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}), \quad d\omega^\alpha = -\sum_\beta \omega^\alpha_{beta} \wedge \omega_\beta, \qquad (2.1)\\
&d\omega^\alpha_\beta = -\sum_\gamma \omega^\alpha_\gamma \wedge \omega^\gamma_\beta + \frac{1}{2}R_{\alpha\beta ij}\omega^i \wedge \omega^j, \\
&R_{\alpha\beta ij} = \sum_k (h^\alpha_{ik}h^\beta_{jl} - h^\alpha_{il}h^\beta_{jk})
\end{aligned}$$

From the condition on the dimensions of M and $\bar{M}^{n+p}_p(c)$ it follows that $\{e_{1*}, \ldots, e_{n*}; e_{(n+1)*}, \ldots, e_{(n+p)*}\}$ is a frame for $T^\perp(M)$. Noticing this, we see that

$$R_{ijkl} = \frac{c}{4}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}) \qquad (2.2)$$

We call $H = \frac{1}{n}\sum_\alpha trh^\alpha$ the mean curvature of M and $S = \sum_{ij\alpha}(h^\alpha_{ij})^2$ the square of the length of the second fundamental form. If H is identically zero then M is said to be maximal. M is totally geodesic if h=0.
From (2.2) we have the Ricci tensor R_{ij} given by

$$R_{ij} = \sum_k R_{ikjk} = \frac{n-1}{4}c\delta_{ij} + \sum_{\alpha k} h^\alpha_{ik}h^\alpha_{kj} \qquad (2.3)$$

Thus the Ricci curvature R is

$$R = R_{ii} = \frac{c}{4}(n-1) + S \tag{2.4}$$

From (2.3) the scalar curvature is given by

$$\rho = \sum_j R_{jj} = \frac{n(n-1)}{4}c + S \tag{2.5}$$

Let h^α_{ijk} denote the covariant derivative of h^α_{ij}. Then we define h^α_{ijk} by

$$\sum_k h^\alpha_{ijk}\omega^k = dh^\alpha_{ij} + \sum_k h^\alpha_{kj}\omega^k_i + \sum_k h^\alpha_{ik}\omega^k_j + \sum_\beta h^\beta_{ij}\omega^\beta_\alpha \tag{2.6}$$

and $h^\alpha_{ijk} = h^\alpha_{ikj}$. Taking the exterior derivative of (2.6) we define the second covariant derivative of h^α_{ij} by

$$\sum_l h^\alpha_{ijkl}\omega^l = dh^\alpha_{ijk} + \sum_l h^\alpha_{ljk}\omega^l_i + \sum_l h^\alpha_{ilk}\omega^l_j + \sum_l h^\alpha_{ijl}\omega^l_k + \sum_\beta h^\beta_{ijk}\omega^\beta_\alpha \tag{2.7}$$

Using (2.7) we obtain the Ricci formula

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{mj}R_{mikl} + \sum_m h^\alpha_{im}R_{mjkl} + \sum_\beta h^\beta_{ij}R_{\beta\alpha kl} \tag{2.8}$$

The Laplacian $\triangle h^\alpha_{ij}$ of the second fundamental form h^α_{ij} is defined as $\triangle h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}$. Therefore,

$$\begin{aligned}\triangle h^\alpha_{ij} =& \frac{c}{4}(n-1)\sum h^\alpha_{ij} + \sum_{\beta mk} h^\alpha_{mi}h^\beta_{mk}h^\beta_{kj} + \sum_{\beta mk} h^\alpha_{km}h^\beta_{mk}h^\beta_{ij} \\ &+ \sum_{\beta mk} h^\beta_{ki}h^\alpha_{jm}h^\beta_{mk} - 2\sum_{\beta mk} h^\beta_{ki}h^\alpha_{mk}h^\beta_{mj}\end{aligned} \tag{2.9}$$

From $\frac{1}{2}\triangle\sum_{\alpha ij}(h^\alpha_{ij})^2 = \sum_{\alpha ijk}(h^\alpha_{ijk})^2 + \sum_{\alpha ij} h^\alpha_{ij}\triangle h^\alpha_{ij}$ we obtain,

$$\begin{aligned}\frac{1}{2}\triangle\sum_{\alpha ij}(h^\alpha_{ij})^2 =& \sum_{\alpha ijk}(h^\alpha_{ijk})^2 + \frac{c}{4}(n-1)\sum_{\alpha ij}(h^\alpha_{ij})^2 + \sum_{\alpha\beta ijkl} h^\alpha_{ij}h^\alpha_{kl}h^\beta_{lk}h^\beta_{ij} \\ &+ \sum_{\alpha\beta ijkl}(h^\alpha_{li}h^\beta_{lj} - h^\beta_{li}h^\alpha_{lj})(h^\alpha_{ki}h^\beta_{kj} - h^\beta_{ki}h^\alpha_{kj})\end{aligned} \tag{2.10}$$

3 Proof of the theorem

Let M be an n-dimensional compact totally real maximal spacelike submanifold isometrically immersed in$\bar{M}_p^{n+p}(c)$. For each α let H_α denote the symmetric matrix (h_{ij}^α) and let $S_{\alpha\beta} = \sum_{ij} h_{ij}^\alpha h_{ij}^\beta$. Then the $(n+2p)\times(n+2p)$-matrix is symmetric and can be assumed to be diagonal for a suitable choice of $e_{n+1},\dots,e_{n+p}$. Setting $S_\alpha = S_{\alpha\alpha} = trH_\alpha^2$ and $S = \sum_\alpha S_\alpha$, equation (2.10) can be rewritten as

$$\begin{aligned}\frac{1}{2}\triangle S &= \sum_{\alpha ijk}(h_{ijk}^\alpha)^2 + \frac{c}{4}(n-1)S + \sum_\alpha S_\alpha^2 + \sum_{\alpha\beta} tr(H_\alpha H_\beta - H_\beta H_\alpha)^2 \\ &= \sum_{\alpha ijk}(h_{ijk}^\alpha)^2 + \frac{c}{4}(n-1)S + \sum_\alpha S_\alpha^2 + \frac{1}{n+2p}S^2 \\ &+ \frac{1}{n+2p}\sum_{\alpha>\beta}(S_\alpha - S_\beta)^2 + \sum_{\alpha\beta} tr(H_\alpha H_\beta - H_\beta H_\alpha)^2 \qquad (3.1) \\ &= \sum_{\alpha ijk}(h_{ijk}^\alpha)^2 + (\frac{c}{4}(n-1) + \frac{1}{n+2p}S)S + \frac{1}{n+2p}\sum_{\alpha>\beta}(S_\alpha - S_\beta)^2 \\ &+ \sum_{\alpha\beta} tr(H_\alpha H_\beta - H_\beta H_\alpha)^2\end{aligned}$$

From (3.1) we see that $\int_M \frac{1}{2}\triangle S dv \geq \int_M \sum_{\alpha ijk}(h_{ijk}^\alpha)^2 dv + \int_M(\frac{c}{4}(n-1) + \frac{1}{n+2p}S)Sdv$ where dv is the volume element of M. By the well known theorem [4], $\triangle S = 0$.

Therefore, $0 \geq \int_M \sum_{\alpha ijk}(h_{ijk}^\alpha)^2 dv + \int_M(\frac{c}{4}(n-1) + \frac{1}{n+2p}S)Sdv$ which implies that

$$\int_M (\frac{c}{4}(n-1) + \frac{1}{n+2p}S)Sdv \leq 0 \qquad (3.2)$$

Thus either $S = 0$ implying M is totally geodesic or $S \leq \frac{(1-n)(n+2p)}{4}c$. This shows that M is totally geodesic for $c \geq 0, n > 1$ or $0 \leq S \leq \frac{(1-n)(n+2p)}{4}c$ for $c < 0, n > 1$. Using equations (2.4) and $S \leq \frac{(1-n)(n+2p)}{4}c$ we see that $R \leq \frac{c}{4}(n-1)(1-n-2p)$ and for $c \geq 0, n > 1$ M is totally geodesic. Similarly, from equations (2.5) and $S \leq \frac{(1-n)(n+2p)}{4}c$ we get $\rho \leq \frac{p(1-n)}{2}c, \quad n > 1, c \geq 0$ which implies that M is totally geodesic. This proves our theorem.

Conclusion

In this manuscript we studied curvature pinching of an n-dimensional compact totally real maximal spacelike submanifold M immersed in an indefinite complex space form $\bar{M}_p^{n+p}(c)$. In conclusion, we have shown that M is totally geodesic if the Ricci curvature R is less than or equal to $\frac{c}{4}(n-1)(1-n-2p), \quad n>1, \quad c\geq 0$. Moreover, if the scalar curvature $\rho \leq \frac{p(1-n)}{2}c, \quad n>1, c\geq 0$, then M is totally geodesic.

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