

## Some Inequalities on Sectional Curvature

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### Abstract

The aim of this paper is to study some inequalities relating the sectional curvatures of indefinite totally real submanifolds with the corresponding curvature of the indefinite complex space form and the size of the second fundamental form of the submanifold.

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## 1 Introduction

Let  $\bar{M}_{s+t}^{n+p}(c)$ ,  $c \neq 0$  be an indefinite complex space form of holomorphic sectional curvature  $c$ , then real dimension of  $\bar{M} = 2n + 2p$  and  $index = 2s + 2t$ , with  $0 \leq s \leq n$  and  $0 \leq t \leq p$ . Let  $J$  be the almost complex structure and  $g$  the metric tensor of  $\bar{M}_{s+t}^{n+p}(c)$  given by

$$g(X, Y) = - \sum_{i=1}^{s+t} X_i Y_i + \sum_{j=s+t+1}^{n+p} X_j Y_j \quad (1.1)$$

Let  $M_s^n$  be a  $2n$ -dimensional indefinite totally real submanifold of index  $2s$  immersed in  $\bar{M}_{s+t}^{n+p}(c)$ . A submanifold  $M$  of a Kaehler manifold is called totally real if each tangent space of  $M$  is mapped into itself by the almost complex structure of the Kaehler manifold [3]. A Kaehler manifold of constant holomorphic sectional curvature is called a complex space form. We choose a local orthonormal frame field  $\{e_1, \dots, e_n; J e_1, \dots, J e_n; e_{n+1}, \dots, e_{n+p}; J e_{n+1}, \dots, J e_{n+p}\}$

on  $\bar{M}_{s+t}^{n+p}$  in such a way that restricted to  $M_s^n$ ;  $e_1, \dots, e_n$ ;  $Je_1, \dots, Je_n$  are tangent to  $M_s^n$  and  $e_{n+1}, \dots, e_{n+p}$ ;  $Je_{n+1}, \dots, Je_{n+p}$  are normal to  $M_s^n$ . Moreover,  $\varepsilon_i = g(e_i, e_i) = g(Je_i, Je_i) = -1$ , when  $1 \leq i \leq s$   
 $\varepsilon_i = g(e_i, e_i) = g(Je_i, Je_i) = 1$ , when  $s+1 \leq i \leq n$   
 $\varepsilon_\alpha = g(e_\alpha, e_\alpha) = g(Je_\alpha, Je_\alpha) = -1$ , when  $n+1 \leq \alpha \leq n+t$   
 $\varepsilon_\alpha = g(e_\alpha, e_\alpha) = g(Je_\alpha, Je_\alpha) = 1$ , when  $n+t+1 \leq \alpha \leq n+p$   
 Let  $\bar{\nabla}$  be the covariant differentiation with respect to  $g$  and  $\bar{\nabla}$  the covariant differentiation induced on from  $g$ . Then the Gauss and Weingarten formulas are

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad \text{and} \quad \bar{\nabla}_X N = -A_N X + \nabla_X^\perp N \quad (1.2)$$

for all  $X, Y \in T(M_s^n)$  and  $N \in T^\perp(M_s^n)$ . Here  $h(X, Y)$  is the second fundamental form of the immersion,  $A_N$  the second fundamental tensor associated with  $N$  and  $\nabla^\perp$  the connection on the normal bundle induced from  $\bar{\nabla}$ . Tensors  $h$  and  $A$  are related by the following equation:

$$h(X, Y) = \sum_{\alpha=n+1}^{n+p} \varepsilon_\alpha g(A_{e_\alpha} X, Y) e_\alpha + \sum_{\alpha=n+1}^{n+p} \varepsilon_\alpha g(A_{Je_\alpha} X, Y) Je_\alpha \quad (1.3)$$

A vector  $X$  is said to be timelike, spacelike or null if  $g(X, X) < 0$ ,  $g(X, X) > 0$ ,  $g(X, X) = 0$  respectively.  $M_s^n$  is defined to be timelike totally geodesic (resp. spacelike totally geodesic) submanifold if  $h(X, U) = 0$  for all  $X, U$  timelike vectors (resp.  $h(Y, V) = 0$  for all  $Y, V$  spacelike vectors).  $M_s^n$  is said to be mixedlike totally geodesic submanifold if  $h(X, Y) = 0$  for all  $X$  timelike and  $Y$  spacelike vectors [9]. By a plane section we mean a 2-dimensional linear subspace of a tangent space. A plane  $P = X, Y$  is called non-degenerate if  $g(X, X)g(Y, Y) - g(X, Y)^2 \neq 0$ . Otherwise it is degenerate. For tangent vectors  $X, Y$  at any point, if  $g(X, X) = g(Y, Y) = 1$  (resp.  $g(X, X) = g(Y, Y) = -1$ ) and  $g(X, Y) = 0$  then the pair  $\{X, Y\}$  is orthonormal of signature  $(+, +)$  (resp.  $(-, -)$ ). The signature of a non-degenerate plane  $P$  is  $(+, +)$ ,  $(-, -)$  or  $(-, +)$  depending on the signature of the restriction of the metric tensor to  $P$  [4]. The curvature tensor  $\bar{R}$  of  $\bar{M}_{s+t}^{n+p}(c)$  is given by [2]

$$\begin{aligned} \bar{R}(X, Y, Z, W) = & \frac{c}{4} [g(X, Z)g(Y, W) - g(X, W)g(Y, Z) + g(X, JZ)g(Y, JW) \\ & - g(X, JW)g(Y, JZ) + 2g(X, JY)g(Z, JW)] \end{aligned} \quad (1.4)$$

where  $X, Y, Z, W$  are vector fields on  $\bar{M}_{s+t}^{n+p}(c)$ . Moreover, the equation of Gauss is given by

$$g(\bar{R}(X, Y)Z, W) = g(R(X, Y)Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)) \tag{1.5}$$

for all vectors  $X, Y, Z, W$  tangent to  $M_s^n$  where  $R$  is the curvature tensor field of  $M_s^n$ . Our main result is:

**Theorem 1.1.** *Let  $M_s^n$  be an indefinite totally real submanifold of an indefinite complex space form  $\bar{M}_{s+t}^{n+p}(c)$ . Then for every holomorphic plane  $P$  of  $M_s^n$ , we have  $\frac{c}{4} \geq K(P) \geq \frac{c}{4} - \frac{1}{2}|h|^2$*

## 2 Sectional curvature

From (1.4) and (1.5) we have the following expression for the curvature tensor of  $M_s^n$ .

$$g(R(X, Y)Z, W) = \frac{c}{4}[g(X, Z)g(Y, W) - g(X, W)g(Y, Z) + g(X, JZ)g(Y, JW) - g(X, JW)g(Y, JZ) + 2g(X, JY)g(Z, JW)] + g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)) \tag{2.1}$$

for vectors  $X, Y, Z, W$  tangent to  $M_s^n$ . Thus the sectional curvature  $K(X \wedge Y)$  of  $M_s^n$  is given by :

$$K(X \wedge Y) = \frac{g(R(X, Y)X, Y)}{Q(X, Y)} = \frac{c}{4} + \frac{3cg(X, JY)^2}{4Q(X, Y)} + \frac{g(h(X, X), h(Y, Y)) - g(h(X, Y), h(X, Y))}{Q(X, Y)} \tag{2.2}$$

where  $Q(X, Y) = g(X, X)g(Y, Y) - g(X, Y)^2 \neq 0$  for all vectors  $X, Y$  tangent to  $M_s^n$ .

**Proposition 2.1.** *Let  $M_s^n$  be an indefinite totally real submanifold of an indefinite complex space form  $\bar{M}_{s+t}^{n+p}(c)$ . Then the mixed sectional curvature  $K(X \wedge Y)$  of  $M_s^n$  is given by  $K(X \wedge Y) = \frac{c}{4} - g(h(X, Y), h(Y, X)) + g(h(X, X), h(Y, Y))$  for all orthonormal  $X$  timelike and  $Y$  spacelike vectors.*

*Proof.* Let  $X$  and  $Y$  be orthonormal timelike and spacelike vectors respectively which span a non-degenerate plane of signature  $(-, +)$ . Then  $Q(X, Y) = g(X, X)g(Y, Y) - g(X, Y)^2 = -1$ . Thus (2, 2) gives

$$\begin{aligned} K(X \wedge Y) &= \frac{g(R(X, Y)X, Y)}{Q(X, Y)} = -g(R(X, Y)X, Y) \\ &= -\frac{c}{4}g(X, X)g(Y, Y) - g(X, Y)g(Y, X) + g(h(X, X), h(Y, Y)) \\ &\quad - g(h(X, Y), h(Y, X)) \\ &= \frac{c}{4} - g(h(X, Y), h(Y, X)) + g(h(X, X), h(Y, Y)). \end{aligned}$$

□

Furthermore, using the definitions of timelike, spacelike and mixedlike totally geodesic submanifolds and by virtue of equation (2.2), we get

**Proposition 2.2.** *Let  $M_s^n$  be a timelike or spacelike totally geodesic indefinite totally real submanifold of an indefinite complex space form  $\bar{M}_{s+t}^{n+p}(c)$ ,  $c \neq 0$ . Then its sectional curvature,  $K(X \wedge U) = \frac{c}{4}$ .*

**Proposition 2.3.** *Let  $M_s^n$  be a timelike or spacelike totally geodesic indefinite totally real submanifold of an indefinite complex space form  $\bar{M}_{s+t}^{n+p}(c)$ ,  $c \neq 0$ . Then its sectional curvature,  $K(X \wedge Y) = \frac{c}{4} + g(h(X, X), h(Y, Y))$ .*

### 3 Proof of the theorem

If  $e_1, \dots, e_n$  is any orthonormal basis of the tangent space of  $M_s^n$ , then the length of  $h$  is defined by  $|h|^2 = \sum_{i,j=1}^n |h(e_i, e_j)|^2$ . Now, let  $e_1, e_2$  be any orthonormal basis of a plane section  $P$  of  $M_s^n$  at any point  $x \in M_s^n$ . Then from the Gauss curvature equation we have

$$K(P) = K(e_1 \wedge e_2) = \frac{c}{4} + \varepsilon_1 \varepsilon_2 g(h(e_1, e_1), h(e_2, e_2)) - |h(e_1, e_2)|^2 \quad (3.1)$$

and the length of  $h$  is  $|h|^2 = \sum_{i,j=1}^2 |h(e_i, e_j)|^2 = |h(e_1, e_1)|^2 + 2|h(e_1, e_2)|^2 + |h(e_2, e_2)|^2$  so that

$$|h(e_1, e_2)|^2 = \frac{1}{2}|h|^2 - \frac{1}{2}|h(e_1, e_1)|^2 - \frac{1}{2}|h(e_2, e_2)|^2 \quad (3.2)$$

Substituting (3.2) into (3.1) and completing the square we get  $K(P) = \frac{c}{4} - \frac{1}{2}|h|^2 + \frac{1}{2}(\varepsilon_1(h(e_1, e_1) + \varepsilon_2 h(e_2, e_2)) - |h(e_1, e_2)|)^2$ . Thus we see that  $\frac{c}{4} \geq K(P) \geq \frac{c}{4} - \frac{1}{2}|h|^2$

## Conclusion

In this paper we studied the geometry of indefinite totally real submanifolds of an indefinite complex space form. Moreover we studied some inequalities relating the sectional curvatures of the submanifold with the corresponding curvature of the indefinite complex space form and the size of the second fundamental form of the submanifold. We have shown that for every plane  $P$  of  $M_s^n$ , we have the inequalities  $\frac{c}{4} \geq K(P) \geq \frac{c}{4} - \frac{1}{2}|h|^2$ .

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