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Mathematical Modeling of Uganda Population Growth

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Abstract

Uganda is a landlocked country in East Africa. It is bordered on the east by Kenya, north by Sudan, west by the Democratic Republic of Congo, southwest by Rwanda and to the south by Tanzania. It has an area of 236,040 square kilometers. The population of Uganda is predominated in the rural with highest density in the southern regions. The purpose of this paper focuses on the application of logistic equation to model the population growth of Uganda using data from 1980 to 2010 (inclusive). The data used were collected from International Data Base (IDB) online and were analyzed by using MATLAB software. We also used least square method to compute the population growth rate, the carrying capacity and the year when the population of Uganda will be approximately a half of the value of its carrying capacity. Population growth of any country depends on the vital coefficients. In the case of Uganda we found that the vital coefficients α and β are 0.0356 and 1.20569×10^{-10} respectively. Thus the population growth rate of Uganda, according to this model, is 3.56% per annum. This approximated population growth rate compares well with statistically predicted values in literature. We also found that the population of

Uganda, 58 years from the year 2010 is expected to be 147633806 while the predicted carrying capacity for the population is 295267612.

Mathematics Subject Classification: 92D25

Keywords: Logistic growth model, Carrying capacity, Vital coefficients, Population growth rate

1 Introduction

Population projection has become one of the most important problems in the world. Population sizes and growth in a country directly influence the situation of economy, policy, culture, education and environment of that country and determine exploring and cost of natural resources. No one wants to wait until those resources are exhausted because of population explosion. Every government and collective sectors always require accurate idea about the future size of various entities like population, resources, demands and consumptions for their planning activities. To obtain this information, the behavior of the connected variables is analyzed based on the previous data by the statisticians and mathematicians at first, and using the conclusions drawn from the analysis, they make future projections of the aimed at variable. There are enormous concerns about the consequences of human population growth for social, environment and economic development. Intensifying all these problems is population growth. Mathematical modeling is a broad interdisciplinary science that uses mathematical and computational techniques to model and elucidate the phenomena arising in life sciences. Thus it is a process of mimicking reality by using the language of mathematics. Many people examine population growth through observation, experimentation or through mathematical modeling. Mathematical models can take many forms, including but not limited to dynamical systems, statistical models and differential equations. These and other types of models can overlap, with a given model involving a variety of abstract structures. In this paper we model the population growth of Uganda by using Verhulst model (logistic growth model). The use of the logistic growth model is widely established in many fields of modeling and forecasting [1]. First order differential equations govern the growth of various species. At first glance it would seem impossible to model the growth of a species by a differential equation since the population of any species always changes by integer amounts. Hence the population of any species can never be a differentiable function of time. However if a given population is very large and it is suddenly increased by one, then the change is very small compared to the given population [4]. Thus we make the assumption that large populations change continuously and even differentiably with time. The projections of future population are normally based on present population. Ideally if the population continues to grow without bound, nature will take over and the death rate will rise to solve the problem. Unfortunately, this is not the most attractive scenario: instead the birth rate would rather be controlled in order to reduce population growth. In this paper, we will determine the carrying capacity and the vital coefficients governing the population growth of Uganda. Farther this paper gives an insight on how to determine the carrying capacity and the vital coefficients, governing population growth, by using the least square method.

2 Methodology

A research is best understood as a process of arriving at dependent solutions to the problems through the systematic collection, analysis and interpretation of data. In relation to this paper, secondary classified yearly population data of Uganda from 1980-2010 (inclusive) were collected from International Data Base (IDB). MATLAB software was used to compute the predicted population values and in plotting down the graphs of actual and predicted population values against time in years. We also used least square method to compute the population growth rate, the carrying capacity and the year when the population of Uganda will be approximately a half of the value of its carrying capacity.

3 Development of the model

Let $P(t)$ denote the population of a given species at time t and let α denote the difference between its birth rate and death rate. If this population is isolated, then $\frac{d}{dt}P(t)$, the rate of change of the population, equals $\alpha P(t)$ where α is a constant that does not change with either time or population. The differential equation governing population growth in this case is

$$\frac{d}{dt}P(t) = \alpha P(t) \tag{1}$$

where, t represents the time period and α, referred to as the Malthusian factor, is the multiple that determines the growth rate. This mathematical model, of population growth, was proposed by an Englishman, Thomas R. Malthus [3], in 1798.
Equation (1) is a non-homogeneous linear first order differential equation known as Malthusian law of population growth. $P(t)$ takes on only integral

values and it is a discontinuous function of t. However, it may be approximated by a continuous and differentiable function as soon as the number of individuals is large enough [7].
The solution of equation (1) is

$$P(t) = P_0 e^{\alpha t} \tag{2}$$

Hence any species satisfying the Malthusian law of population growth grows exponentially with time. This model is often referred to as *The Exponential Law* and is widely regarded in the field of population ecology as the first principle of population Dynamics. At best, it can be described as an approximate physical law as it is generally acknowledged that nothing can grow at a constant rate indefinitely. As population increases in size, the environment's ability to support the population decreases. As the population increases per capita food availability decreases, waste products may accumulate and birth rates tend to decline while death rates tend to increase. Thus it seems reasonable to consider a mathematical model which explicitly incorporates the idea of carrying capacity (limiting value). A Belgian Mathematician Verhulst [5], showed that the population growth not only depends on the population size but also on how far this size is from its upper limit i.e. its carrying capacity (maximum supportable population). He modified Malthus's Model to make the population size proportional to both the previous population and a new term

$$\frac{\alpha - \beta P(t)}{\alpha} \tag{3}$$

where α and β are called the vital coefficients of the population. This term reflects how far the population is from its maximum limit. However, as the population value grows and gets closer to $\frac{\alpha}{\beta}$, this new term will become very small and tend to zero, providing the right feedback to limit the population growth. Thus the second term models the competition for available resources, which tends to limit the population growth. So the modified equation using this new term is:

$$\frac{d}{dt}P(t) = \frac{\alpha P(t)(\alpha - \beta P(t))}{\alpha} \tag{4}$$

This is a nonlinear differential equation unlike equation (1) in the sense that one cannot simply multiply the previous population by a factor. In this case the population $P(t)$ on the right of equation (4) is being multiplied by itself. This equation is known as the logistic law of population growth. Putting $P = P_0$ for $t = 0$, where P_0 represents the population at some specified time, $t = 0$, equation (4) becomes

$$\frac{d}{dt}P = \alpha P - \beta P^2. \tag{5}$$

Separating the variables in equation (5) and integrating, we obtain $\int \frac{1}{\alpha}(\frac{1}{P} + \frac{\beta}{\alpha - \beta P})dP = t + c$, so that

$$\frac{1}{\alpha}(\log P - \log(\alpha - \beta P)) = t + c. \tag{6}$$

Using $t = 0$ and $P = P_0$ we see that $c = \frac{1}{\alpha}(\log P_0 - \log(\alpha - \beta P_0))$. Equation (6) becomes

$\frac{1}{\alpha}(\log P - \log(\alpha - \beta P)) = t + \frac{1}{\alpha}(\log P_0 - \log(\alpha - \beta P_0))$. Solving for P yields

$$N = \frac{\frac{\alpha}{\beta}}{1 + (\frac{\frac{\alpha}{\beta}}{P_0} - 1)e^{-\alpha t}} \tag{7}$$

If we take the limit of equation (7) as $t \to \infty$, we get (since $\alpha > 0$)

$$P_{max} = \lim_{t \to \infty} P = \frac{\alpha}{\beta} \tag{8}$$

Next, we determine the values of α, β and P_{max} by using the least square method. Differentiating equation (7), twice with respect to t gives

$$\frac{d^2P}{dt^2} = \frac{C\alpha^3 e^{\alpha t}(C - e^{\alpha t})}{\beta(C + e^{\alpha t})^3} \tag{9}$$

where $C = \frac{\frac{\alpha}{\beta}}{P_0} - 1$.
At the point of inflection this second derivative of P must be equal to zero. This will be so, when

$$C = e^{\alpha t} \tag{10}$$

Solving for t in equation (10) gives

$$t = \frac{lnC}{\alpha} \tag{11}$$

This is the time when the point of inflection occurs, that is, when the population is a half of the value of its carrying capacity. Let the time when the point of inflexion occurs be $t = t_k$. Then $C = e^{\alpha t}$ becomes $C = e^{\alpha t_k}$. Using this new value of C and replacing $\frac{\alpha}{\beta}$ by K equation (7) becomes

$$P = \frac{K}{1 + e^{-\alpha(t - t_k)}} \tag{12}$$

Let the coordinates of the actual population values be (t, p) and the coordinates of the predicted population values with the same abscissa on the fitted curve be (t, P). Then the error in this case is given by $(P - p)$. Since some of the actual population data points lie below the curve of predicted values while others lie above it, we square $(P - p)$ to ensure that the error is positive. Thus, the total squared error, e, in fitting the curve is given by

$$e = \sum_{i=1}^{n} (P_i - p_i)^2 \tag{13}$$

Equation (13) contains three parameters K, α and t_k. To eliminate K we let

$$P = Kh \tag{14}$$

where

$$h = \frac{1}{1 + e^{-\alpha(t-t_k)}} \tag{15}$$

Using the value of P in equation (14) and algebraic properties of inner product to equation (13), we have

$$\begin{aligned} e &= \sum_{i=1}^{n} (P_i - p_i)^2 \\ &= (P_1 - p_1)^2 + \cdots + (P_n - p_n)^2 \\ &= (Kh_1 - p_1)^2 + \cdots + (Kh_n - p_n)^2 \\ &= \mid (Kh_1 - p_1, \ldots, Kh_n - p_n) \mid^2 \\ &= \mid (Kh_1 \ldots Kh_n) - (p_1, \ldots, p_n) \mid^2 \\ &= \mid KH - W \mid^2 \\ &= < KH - W, KH - W > \\ &= K^2 < H, H > -2K < H, W > + < W, W > \end{aligned}$$

where $H = (h_1, \ldots, h_n)$ and $W = (p_1, \ldots, p_n)$. Thus,

$$e = K^2 < H, H > -2K < H, W > + < W, W > \tag{16}$$

Taking partial derivative of e with respect to K and equating it to zero, we obtain $2K < H, H > -2 < H.W >= 0$. This gives

$$K = \frac{< H, W >}{< H, H >} \tag{17}$$

Substituting this value of K into equation (16), we get

$$e =< W, W > - \frac{< H, W >^2}{< H, H >} \tag{18}$$

This equation is converted into an error function, MATLAB program, [appendix 2(a)], that contains just two parameters, α and t_k. Their values were found, MATLAB program, [appendix 2(b)] and used in equation (17) to find the value of K, MATLAB program, [appendix 2(c)].

4 Results

Table 1: Actual values of population

Year	Actual Population		Year	Actual Population
1980	12414719		1996	21248718
1981	12725252		1997	21861011
1982	13078930		1998	22502140
1983	13470393		1999	23227669
1984	13919514		2000	23955822
1985	14391743		2001	24690002
1986	14910724		2002	25469579
1987	15520093		2003	26321962
1988	16176418		2004	27233661
1989	16832384		2005	28199390
1990	17455758		2006	29206503
1991	18082137		2007	30262610
1992	18729453		2008	31367972
1993	19424376		2009	32369558
1994	20127590		2010	33398682
1995	20689516			

Using actual population values, their corresponding years from table 1 and MATLAB programs, [appendix 2(a) and 2(b)], we find that the values of α and t_k are 0.0356 and 2068 respectively. Thus, the value of the population growth rate of Uganda is approximately 3.56% per annum while the population will be a half of its limiting value in the year 2068. Using values of α and t_k and MATLAB program [appendix 2(c)], equation (17) gives

$$K = P_{max} = 295267612 \tag{19}$$

This is the predicted carrying capacity or limiting value of the population of Uganda. Using equations (8), we find that

$$\beta = \frac{0.0356}{295267612} = 1.20569 \times 10^{-10} \tag{20}$$

This value is the other vital coefficient of the population. If we let t=0 to correspond to the year 1980, then the initial population will be $P_0 = 12414719$. Substituting the values of P_0, $\frac{\alpha}{\beta}$ and α into equation (7), we obtain

$$P = \frac{295267612}{1 + (22.78367259) \times (0.965)^t} \tag{21}$$

This equation was used to compute the predicted values of the population. Using values of α, β and P_0 equation (11) gives the time at the point of inflection to be

$$t \approx 88 \tag{22}$$

which, when added to the actual year corresponding to t=0 i.e. 1980, gives 2068 as earlier found as the value of t_k. Using this value of t and equation (21), we get

$$\frac{\alpha}{2\beta} = 147633806 \tag{23}$$

Thus, the population of Uganda is predicted to be 147633806 in the year 2068. This predicted population value is a half of its carrying capacity. The following table contains the predicted population values and their corresponding actual population values.

Table 2: Actual and predicted values of population

Year	Actual pop.	Predicted pop.		Year	Actual pop.	Predicted pop.
1980	12414719	12414719		1996	21248718	21266298
1981	12725252	12845405		1997	21861011	21980197
1982	13078930	13290330		1998	22502140	22716074
1983	13470393	13749915		1999	23227669	23474471
1984	13919514	14224592		2000	23955822	24255934
1985	14391743	14714799		2001	24690002	25061014
1986	14910724	15220984		2002	25469579	25890262
1987	15520093	15743605		2003	26321962	26744234
1988	16176418	16283127		2004	27233661	27623485
1989	16832384	16840024		2005	28199390	28528571
1990	17455758	17414779		2006	29206503	29460048
1991	18082137	18007881		2007	30262610	30418471
1992	18729453	18619830		2008	31367972	31404390
1993	19424376	19251129		2009	32369558	32418352
1994	20127590	19902293		2010	33398682	33460902
1995	20689516	20573841				

Below is the graph of actual and predicted population values against time.

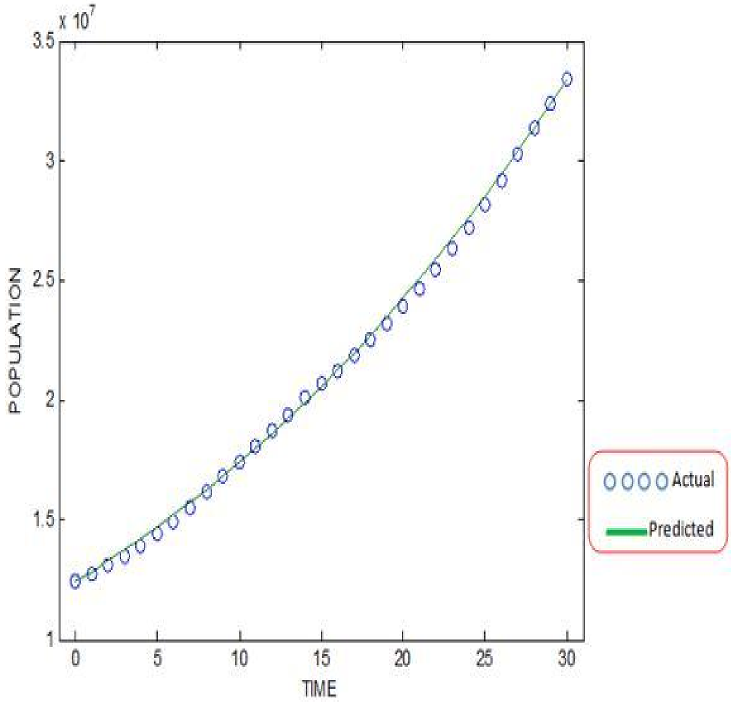

Figure 1: Graph of actual and predicted population values against time.

The following is the graph of predicted population values against time. The values were computed using equation (21).

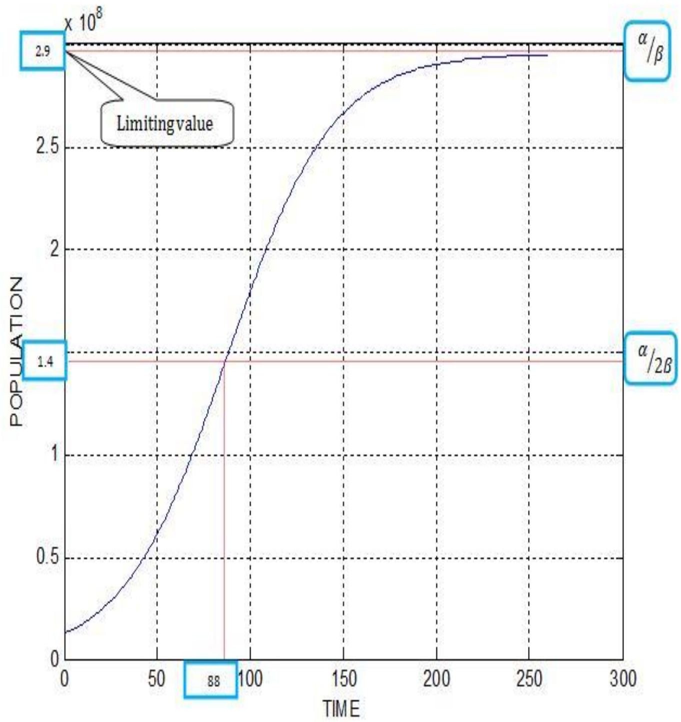

Figure 2: Graph of predicted population values against time.

5 Discussion

From figure 1, we see that the actual data points and predicted values are very close to one another. This indicates that the error between them is very small. In figure 2, we can see that the graph of the predicted population values is an S-shaped curve. This shows that the values fitted well into the logistic curve. At first, the population starts to grow going through an exponential growth phase reaching 147633806 (a half of its carrying capacity) in the year 2068 after which the rate of growth is expected to slow down. As it gets closer to the carrying capacity, 295267612, the growth is again expected to drastically slow down and reach a stable level. The population growth rate of Uganda according to information in International Data Base (IDB) was approximately 3.3% in 2002, 2003, 3.5% in 2004, 2005, 2006 and 3.6% in the years 2007, 2008 and 2010 which corresponds well with the findings in this research work of a growth rate of approximately 3.56% per annum.

6 Conclusion

In conclusion we found that the predicted carrying capacity for the population of Uganda is 295267612. Population growth of any country depends also on the vital coefficients. In the case of Uganda we found out that the vital coefficients α and β are 0.0356 and 1.20569×10^{-10} respectively. Thus the population growth rate of Uganda, according to this model, is 3.56% per annum. This approximated population growth rate compares well with the statistically predicted values in literature. Based on this model we also found out that the population of Uganda is expected to be 147633806 (a half of its carrying capacity) in the year 2068. The following are some recommendations: Technological developments, pollution and social trends have significant influence on the vital coefficients α and β, therefore, they must be re-evaluated every few years to enhance the determination of variations in the population growth rate. In order to derive more benefits from models of population growth, one should subdivide populations into different age groups for effective capture, analyses and planning purposes. Other models can be developed by subdividing the population into males and females, since the reproduction rate in a population usually depends more on the number of females than on the number of males. The government should work towards industrialization of the country. This will have an effect in improving its absorptive capacity for development through population growth rate measures. The more industrialized a Nation is, the more living space and food it has and the smaller the coefficient β, thus, raising the carrying capacity. Finally, the government of Uganda should also step up the dissemination of civic education on birth control methods to enable it to manage its resources allocation through efficient and effective population growth rate management principles. However, present attempts appear to provide acceptable predictions for the Uganda population growth.

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Appendix

Map of Uganda

Figure 3: Map of Uganda

Matlab programs

(a) MATLAB program used to find the error function

```
  syms r tk;
t=[1980,1981,1982,1983,1984,1985,1986,1987,1988,1989,1990,1991,
1992,1993,1994,1995,1996,1997,1998,1999,2000,2001,2002,2003,
2004,2005,2006,2007,2008,2009,2010] ;
p=[12414719,12725252,13078930,13470393,13919514,14391743,
14910724,15520093,16176418,16832384,17455758,18082137,
18729453,19424376,20127590,20689516,21248718,21861011,
22502140,23227669,23955822,24690002,25469579,26321962,
27233661,28199390,29206503,30262610,31367972,32369558,33398682] ;
H=1./(1+exp(-r*(t-tk)))
t=t'
p=p'
H=H'
e=(p'*p)-((H'*p)^2/(H'*H))
```

Note that r and tk are replacing α and t_k respectively while t and p are time and actual population respectively.

(b) MATLAB program used to find the values of r and tk which minimize error function

```
banana=@(x)e;
format long
[x]=fminsearch(banana,[0.1,2100])
```

Note that r and tk, in error function, e, must be replaced by x(1) and x(2) respectively. 0.1 and 2100 are starting points.

(c) MATLAB program used to find the value of K

```
r=0.0356;
tk=2068;
t=[1980,1981,1982,1983,1984,1985,1986,1987,1988,1989,1990,1991,
1992,1993,1994,1995,1996,1997,1998,1999,2000,2001,2002,2003,
2004,2005,2006,2007,2008,2009,2010];
p=[12414719,12725252,13078930,13470393,13919514,14391743,
14910724,15520093,16176418,16832384,17455758,18082137,
18729453,19424376,20127590,20689516,21248718,21861011,
```

```
22502140,23227669,23955822,24690002,25469579,26321962,
27233661,28199390,29206503,30262610,31367972,32369558,33398682];
H=1./(1+exp(-r*(t-tk)))
H=H'
p=p'
K=(H'*p)/(H'*H)
```

t and p are time and actual population respectively.

(d) MATLAB program used to find the predicted values

```
t=0 :30 ;
format long
P= 295267612./(1+(22.78367259)*(0.965).^t )
```

t and P represent time and predicted population respectively.

(e) MATLAB program used to plot the graph of actual and predicted values

```
t=0:30;
p=[12414719,12725252,13078930,13470393,13919514,14391743,
14910724,15520093,16176418,16832384,17455758,18082137,
18729453,19424376,20127590,20689516,21248718,21861011,
22502140,23227669,23955822,24690002,25469579,26321962,
27233661,28199390,29206503,30262610,31367972,32369558,33398682];
format long
P= 295267612./(1+(22.78367259)*(0.965).^t )
plot(t,p,'o',t,P)
xlabel('TIME')
ylabel('POPULATION')
```

t, p and P represent time, actual population and predicted population respectively.

(f) MATLAB program used to plot the graph of predicted values with extended time

```
t=0:260;
format long
P= 295267612./(1+(22.78367259)*(0.965).^t )
plot(t,P)
xlabel('TIME')
ylabel('POPULATION')
```

t and P are time and predicted population respectively.

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