Int. J. Adv. Appl. Math. and Mech. 10(3) (2023) 52 – 65 (ISSN: 2347-2529)

Journal homepage: www.ijaamm.com

International Journal of Advances in Applied Mathematics and Mechanics

A novel model for female population on the effects of African Stalk Borer on Saccharum officinarum L. under the sterile Insect Technology Interventions

Research Article

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Received 05 January 2023; accepted (in revised version) 26 January 2023

Abstract: Sugarcane is an important plant, not only for its economic value but its ecological importance. An infestation of E. Walker *Lepidoptera pyralidae*, which occurs naturally in wetland habitats and tall grasses, ravages the sugarcane stalk reducing its value. The study proposes a novel model for formulating the dynamics of the E. walker population with SIT. A mathematical analysis of the proposed governing equations representing the model has unique and positive solutions. A basic reproduction number computed based n wild free equilibrium (WFE) points was found to be $\mathscr{R}_0 < 1$ indicating an eminent wipeout of the wild E. walker population under SIT. The local stability of the WFE indicated that the established $\mathscr{R}_0$ was locally asymptotically stable and $\mathscr{R}_0 < 1$. The global stability showed that WFE is globally asymptotically stable when $\mathscr{R}_0 < 1$. The numerical simulation revealed that the wild E. walker population under Sterile Insect Technology (SIT) will be wiped out after more than 120 weeks, which is unrealistic, considering that the sugarcane matures after approximately 78 weeks. Elasticity analysis of the model parameters based on $\mathscr{R}_0$ indicated that a possible control lies in controlling the eggs laid and sex ratio. The effectiveness of the control is indicated in the numerical simulation that showed that the population of the wild E. walker is wiped out after approximately 130 weeks. Future studies into the area need to refocus on the timelines to investigate other strategies to reduce the wild E. walker population below the sugarcane maturity stage.

MSC: 34D20 • 49J15

Keywords: E. Walker • Sterile Insect Technology • Wild Free Equilibrium

1. Introduction

Sugarcane (*Saccharum officinarum L.*) is a tall grass of the Gramineae family has a stalk that consists of sections approximately 13 cm in length, with each section comprising of a node and an internode [1]. The stalks have a diameter of about 2 cm, and the cane plant has a height of up to 6 m with high, green leaves upwards and dead leaves on the lower parts of the stalks. The node of the stalk has a bud from which a young plant emerges. Sugarcane does well in tropical climates and fertile soils. However, an infestation of E. Walker *Lepidoptera pyralidae*, which occurs naturally in wetland habitats and tall grasses, ravages the sugarcane stalk reducing the farmers' income [2, 3].

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For a long period, E. Walker control methods have included chemical control, crop management, varietal resistance, biological control [4] and, more recently, Sterile Insect Technique (SIT). The SIT method relies on releasing sterilized male insects to the wild insect population. After mating with sterilized males, the wild females will lay eggs that don't hatch into larvae pests. The result is the subsequent suppression of the wild population. The extent to which this is achieved mathematically has remained elusive. Numerous researchers have employed the differential equations approaches to assume that the spread of insects and their management through SIT can be modelled via ordinary differential equations (ODE) within the disease dynamic concept [5–8]. The current paper proposes a novel model for formulating the dynamics of the E. walker population with SIT. The mathematical analysis is performed with a focus on the feasibility of the model in real-life applications. The equilibrium points are determined, and their stability analysis is performed. Numerical simulation is performed to estimate parameter values to depict the population dynamics. Optimal control is also determined, and its existence is established. We also characterize the controls based on Pontryagin's maximum principle. Finally, a numerical simulation for the control is performed to establish the effectiveness of the control in the population dynamics of E. walker.

2. The Model

2.1. The model formulation

The model presented in Figure 1 shows the population fertile male M_f of E. Walker is increased by $\sigma(1-\alpha)\frac{F_f}{K}$ where α is the primary sex ratio, σ is the the number of eggs laid per day on the subsequent population of the wild pests, K is the egg carrying capacity of the wild fertile female F_f. This population is reduced by natural deaths rate μ_M. The population of F_f is increased by rate of female eggs hatched per day $\sigma\alpha$ and competition of mating between wild male and released sterile male $\frac{\beta}{M_f+M_s}$. The population is decreased by natural deaths rate μ_F. The population of sterile males is increased by the release rate r and decrease by natural deaths rate μ_{M_S}.

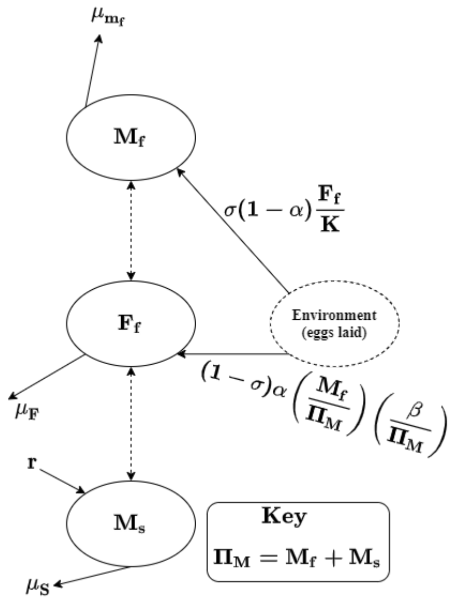

Fig. 1. A model diagram representing the E. walker control where the eggs laid by wild female is controlled by releasing the sterile male into the wild to compete in mating with wild male in order to reduce the population of both wild female and male. The wild female mate with the sterile male to lay sterile eggs.

2.2. The governing equations

The system of equations representing based on the population growth on the Malthus's model for the individual compartments is given by;

$$\frac{dM_f}{dt} = \sigma(1-\alpha)\frac{F_f}{K} - \mu_M M_f \tag{1}$$
$$\frac{dF_f}{dt} = \sigma\alpha\frac{M_f}{M_f+M_s}\frac{\beta}{M_f+M_s} - \mu_F F_f \tag{2}$$
$$\frac{dM_s}{dt} = r - \mu_s M_S \tag{3}$$

The initial conditions governing the model is given by:

$$M_f(0) = M_f^0 > 0, F_f(0) = F_f^0 \geq 0, \;\&\; M_s(0) = M_s^0 > 0 \tag{4}$$

3. Well-posedness of the system

It is shown that the model represented by (1)-(3) has a unique, non-negative solution that exists and is bounded and is feasible in real life.

If we let $X(t) = \left(M_f(t), F_f(t), M_s(t)\right)$ and $f : X \to M_f^{'}$ such that $f = \left(f_1, f_2, f_3\right)$, where

$$\begin{aligned} f_1(X) &= \sigma(1-\alpha)\frac{F_f}{K} - \mu_M M_f \\ f_2(X) &= \sigma\alpha\frac{M_f}{M_f+M_s}\frac{\beta}{M_f+M_s} - \mu_F F_f \\ f_3(X) &= r - \mu_s M_s. \end{aligned} \tag{5}$$

So that we can write (1)-(3) as:

$$X^{'} = f(X(t)); \;\; X(0) = \left(M_{f_0}, F_{f_0}, M_{s_0}\right). \tag{6}$$

Theorem 3.1.
Suppose $f(X(t))$ as given by (6) and the initial condition $X(0) = E = M_f + F_f + M_s > 0$, is non-negative, then system (1)-(3) has a unique solution that is non-negative and bounded.

Proof. f_i are continuous functions and $\frac{\partial f_i}{\partial X_j}$, $1 \leq i, \; j \leq 3$ exist and are continuous functions such that $f(X(t))$ is locally Lipschitz continuous. $X(0) = E = M_f + F_f + M_s > 0$, thus at least one compartment is non-empty. Therefore, there exists a unique solution $X(t)$ of the system, defined in some time interval containing $t = 0$. Let t_0 be the smallest t such that $M_f(0) = 0$ or $F_f(0) = 0$ or $M_s(0)$. By continuity of $M_f(t)$, $F_f(t)$, and $M_s(t)$, $\exists\; t^* > t_0$ such that if $M_f(0) = 0$, then from (1) we get $\frac{dM_f}{dt} = \sigma(1-\alpha)\frac{F_f}{K} \geq 0 \;\forall t \in [t_0, t^*]$. Thus, M_f is increasing function on the time interval $[t_0, t^*]$, $\implies M_f(t) \geq 0 \;\forall t \in [t_0, t^*]$. Similarly, from (2), $\frac{dF_f}{dt} = \sigma\alpha\frac{M_f}{M_f+M_s}\frac{\beta}{M_f+M_s} \geq 0 \;\forall t \in [t_0, t^*]$. Thus, F_f is increasing function on the time interval $[t_0, t^*]$, $\implies F_f(t) \geq 0 \;\forall t \in [t_0, t^*]$. Similarly, if $M_s(t_0) = 0$, then from (3) we obtain $\frac{dM_s}{dt} = r \geq 0 \implies M_s(t) \geq 0 \;\forall \in [t_0, t^*]$. Thus, $E(0) = M_f(0) + F_f(0) + M_s(0) \geq 0$, thus, a solution to the system is non-negative. We use the dissipativity condition of theorem 2.3.6 of [9] to establish the a unique solution exist globally.

$$\begin{aligned} & f(X).X = \left(f_1, f_2, f_3\right).\left(M_f, F_f, M_s\right) \\ & = M_f f_1 + F_f f_2 + M_s f_3 \\ & \leq M_f \sigma(1-\alpha)\frac{F_f}{K} - \mu_M M_f^2 + \sigma\alpha\frac{F_f M_f}{M_f+M_s}\frac{\beta}{M_f+M_s} \\ & -\mu_F F_f^2 + r M_s - \mu_s M_s^2. \end{aligned} \tag{7}$$

Thus a unique solution $X(t)$ exists defined for all $t \geq 0$; such that $M_f \leq E$, $F_f \geq E$, and $M_s \geq E$: where $E = M_f + F_f + M_s$.

□

4. Equilibrium points

The equilibrium points is where there is no change in the variables with time, that is $\frac{dM_f}{dt} = \frac{dF_f}{dt} = \frac{dM_s}{dt} = 0$. We use (3) to obtain

$$M_{s_0} = \frac{r}{\mu_s} \tag{8}$$

We use (8) into (2) to get

$$F_{f_0} = \frac{M^*_{f_0}\beta\sigma(1-\alpha)}{\mu_F} \tag{9}$$

We use (9) to obtain

$$M_{f_0} = 0 \tag{10}$$

We substitute (10) into (9) to get

$$F_{f_0} = 0 \tag{11}$$

Thus the case of $M_{f_0} = 0, F_{f_0} = 0$ and M_{s_0} gives the wild free equilibrium (WFE) $X_0 = \left(0, 0, \frac{r}{\mu_S}\right) \in \Omega\{(M_f, F_f, M_s) \in \mathbb{R}^2_+ | M_f + F_f + M_s| = E$.

4.1. The Control Reproduction Number

We use the next generation matrix to compute the basic reproduction number. We consider M_f and F_f. If we let $x = (M_f, F_f)^T$, thus we set $\mathscr{F} = \left(\mathscr{F}_{M_f}, \mathscr{F}_{F_f}\right)$ and $\mathscr{V} = \left(\mathscr{V}_{M_f}, \mathscr{V}_{F_f}\right)$, where $\mathscr{F}_j,\ j = M_f, F_f$ is the rate of appearance of new wild population in compartment j. $\mathscr{V}_j = \mathscr{V}_j^- - \mathscr{V}_j^+$, where $\mathscr{V}_j^-$ is the rate of transfer out of compartment j and $\mathscr{V}_j^+$ is the rate of transfer into compartment j, Thus, we have

$$\mathscr{F} = \begin{pmatrix} \sigma\alpha\frac{F_f}{K} \\ \sigma\alpha\frac{M_f}{M_f+M_s}\frac{beta}{M_f+M_s} \end{pmatrix}. \tag{12}$$

$$\mathscr{V} = \begin{pmatrix} \mu_M M_f \\ \mu_F F_f \end{pmatrix}. \tag{13}$$

We assume that X_0 is the WFE for the model, we have

$$F = \left[\frac{\partial \mathscr{F}_i}{\partial x_j}(X_0)\right] = \begin{pmatrix} 0 & \frac{\sigma\alpha}{K} \\ \frac{\alpha\sigma\beta\mu_S^2}{r^2} & 0 \end{pmatrix}. \tag{14}$$

$$V = \left[\frac{\partial \mathscr{V}_i}{\partial x_j}(X_0)\right] = \begin{pmatrix} \mu_M & 0 \\ 0 & \mu_F \end{pmatrix}. \tag{15}$$

we compute the inverse of V gives

$$V^{-1} = \begin{pmatrix} \frac{1}{\mu_M} & 0 \\ 0 & \frac{1}{\mu_F} \end{pmatrix}. \tag{16}$$

We use (16) and (15) to obtain

$$FV^{-1} = \begin{pmatrix} 0 & \frac{\alpha\sigma}{K\mu_F} \\ \frac{\alpha\beta\mu_S^2\sigma}{\mu_M r^2} & 0 \end{pmatrix}. \tag{17}$$

The control reproduction number $\mathscr{R}_0$ is given by the spectral radius of FV^{-1}, thus

$$\mathscr{R}_0 = \frac{\alpha\mu_s\sigma}{r}\sqrt{\frac{\beta}{K\mu_M\mu_F}} \tag{18}$$

The control reproduction number is the expected number of wild population produced by a single fertile mating in a completely fertile environment.

4.2. Stability of WFE

4.2.1. Local Stability of WFE

Theorem 4.1.
Given that $X_0 = \left(0, 0, \frac{r}{\mu_S}\right)$ *is a WFE for the model, then* X_0 *is locally asymptotically stable if* $\mathscr{R}_0 < 1$ *and unstable if* $\mathscr{R}_0 > 1$.

Proof. We follow Theorem 2 in [10] as follows;

$$J = \begin{pmatrix} -\mu_M & \frac{\alpha\sigma}{K} & 0 \\ \beta\sigma(1-\alpha) & -\mu_F & 0 \\ 0 & 0 & -\mu_S \end{pmatrix}. \tag{19}$$

The eigenvalues of J is given by

$$\lambda_i = \begin{pmatrix} -\mu_S \\ -\frac{\mu_F}{2} - \frac{\mu_M}{2} - \Pi_\lambda \\ \Pi_\lambda - \frac{\mu_M}{2} - \frac{\mu_F}{2} \end{pmatrix} \tag{20}$$

where $\Pi_\lambda = \left(\frac{-4\beta\alpha^2\sigma^2 + 4\beta\alpha\sigma^2 + K\mu_F^2 - 2K\mu_F\mu_M + K\mu_M^2}{4K}\right)^{\frac{1}{2}}$. λ_i are real parts, hence $\mathscr{R}_0$ is locally asymptotically stable and $\mathscr{R}_0 < 1$.

□

4.2.2. The global stability of WFE

Theorem 4.2.
If $\mathscr{R}_0 \geq 1$, *the WFE of system* (1)-(3) *is globally asymptotically stable in* $\Omega = \{M_f, F_f, M_s\} \in \mathbb{R}_+^3 | M_f \geq 0, F_f \geq 0, M_s \geq 0, M_f + F_f + M_s = E$. *If* $_0 > 1$, *the WFE is unstable.*

Proof. We use the matrix theoretic method by [11] based on the dynamics of the *E. walker* compartment population model as;

$$\frac{dx}{dt} = \mathscr{F}(x,y) - \mathscr{V}(x,y) \tag{21}$$

where $x = (M_f, F_f)^T$, $y = M_s$, and $\mathscr{F}$ and $\mathscr{V}$ are given by (12) and (13), respectively. We can re-write the dynamics of the x (wild compartments) as

$$\frac{dx}{dt} = (F - V)x - f(x,y) \tag{22}$$

where F and V are given by (14) and (15) respectively, and

$$\begin{aligned} f(x,y) &= (F-V)x - \mathscr{F}(x,y) + \mathscr{V}(x,y) \\ &= \begin{pmatrix} 0 \\ \frac{M_f\alpha\beta\mu_S^2\sigma}{r^2} - \frac{M_f\alpha\beta\sigma}{(M_f+M_s)^2} \end{pmatrix}. \end{aligned} \tag{23}$$

and

$$V^{-1}F = \begin{pmatrix} 0 & \frac{\alpha\sigma}{(K\mu_M)} \\ \frac{\alpha\beta\mu_S^2\sigma}{(\mu_F r^2)} & 0 \end{pmatrix}. \tag{24}$$

Eq. (23) indicate that $f(x,y) \geq 0$ in $\Omega \supset \mathbb{R}_+^3$, $F \geq 0$, and $V^{-1} \geq 0$. $V^{-1}F$ is reducible thus conclusion of Theorem 2.2 of [11]. Therefore, we use Theorem 2.1 of [11] to construct the Lyapunov function for the system (1)-(3).

Suppose we let $w^T = (w_1, w_2)$ be the left eigenvector of $V^{-1}F$ corresponding to $\rho(FV^{-1}) = \rho(V^{-1}F) = \mathscr{R}_0$. Thus, we have

$$(w_1, w_2)V^{-1}F = \mathscr{R}_0(w_1, w_2) \tag{25}$$

We substitute the value of $\mathscr{R}_0$ from (18) and simplify to get

$$w_1 = \frac{K^{\frac{1}{2}} \beta^{\frac{1}{2}} \mu_S \mu_M^{\frac{1}{2}} w_2}{\mu_F^{\frac{1}{2}} r}. \tag{26}$$

Suppose we assume $w_2 = z$, were z is a parameter. If we let $z = 1$, then $(w_1, w_2) = (\frac{K^{\frac{1}{2}} \beta^{\frac{1}{2}} \mu_S \mu_M^{\frac{1}{2}} w_2}{\mu_F^{\frac{1}{2}} r}, 1)$, Thus, function Q is given by

$$Q = w^T V^{-1} x = \frac{F_f}{\mu_F} + \frac{K^{\frac{1}{2}} M_f \beta^{\frac{1}{2}} \mu_s w_2}{\mu_F^{\frac{1}{2}} \mu_M^{\frac{1}{2}} r} \tag{27}$$

We differentiate Q along the solution of the (1)-(3) to obtain;

$$\begin{aligned} Q' = & \ (\mathscr{R}_0 - 1) w^T x - w^T V^{-1} f(x, y) \\ & = (\mathscr{R}_0 - 1)\Big(F_f + \frac{K^{\frac{1}{2}} M_f \beta^{\frac{1}{2}} \mu_S \mu_M^{\frac{1}{2}}}{\mu_F^{\frac{1}{2}} r}\Big) \\ & + \frac{\alpha\sigma\beta M_f}{\mu_F}\Big[\frac{1}{(M_f + M_s)^2} - \frac{\mu_S^2}{r^2}\Big] \end{aligned}. \tag{28}$$

If $\mathscr{R}_0 < 1$, the $Q' \leq 0$ in Ω. Thus, Q is the Lyapunov function for the system (1)-(3). We use LaSalle's invariance principle [12] to prove the global stability of WFE as follows: For $Q' = 0 \implies$

$$\begin{aligned} & \frac{\alpha\sigma\beta M_f}{\mu_F}\Big[\frac{\mu_S^2}{r^2} - \frac{1}{(M_f + M_s)^2}\Big] \\ & = (\mathscr{R}_0 - 1)\Big(F_f + \frac{K^{\frac{1}{2}} M_f \beta^{\frac{1}{2}} \mu_S \mu_M^{\frac{1}{2}}}{\mu_F^{\frac{1}{2}} r}\Big) \end{aligned} \tag{29}$$

Since $\mathscr{R}_0 \leq 1$, we get

$$\frac{\alpha\sigma\beta}{\mu_F}\Big[\frac{\mu_S^2 M_f}{r^2} - \frac{M_f}{(M_f + M_s)^2}\Big] \leq 0. \tag{30}$$

Since $\frac{\alpha\sigma\beta}{\mu_F} > 0$, we have

$$\begin{aligned} & \Big[\frac{\mu_S^2 M_f}{r^2} - \frac{M_f}{(M_f + M_s)^2}\Big] \leq 0 \\ & \implies \frac{\mu_S^2}{r^2} \leq \frac{1}{(M_f + M_s)^2} \\ & \implies \frac{\mu_S^2}{r^2} - \frac{1}{(M_f + M_s)^2} \leq 0 \\ & \implies E \leq (M_f + M_s) \end{aligned} \tag{31}$$

The results indicate that $E \leq M_f$, which implies that $E = M_s + M_f$, implying that $F_f = 0$ since $M_f + F_f + M_s = E$. Thus, $(0, 0, \frac{r}{\mu_S})$ is the only invariant set in Ω satifying that $Q' = 0$ when $\mathscr{R}_0 < 1$. Thus, by LaSalle's invariance principle, the WFE is globally asymptotically stable in Ω when $\mathscr{R}_0 < 1$.

When $\mathscr{R}_0 = 1$, then the first term of (28) becomes zero, and $Q' \leq 0$ in Ω. $Q' = 0 \implies$

$$\begin{aligned} Q' = & \ -w^T V^{-1} f(x, y) && = 0 \\ & \frac{\alpha\sigma\beta M_f}{\mu_F}\Big[\frac{1}{(M_f + M_s)^2} - \frac{\mu_S^2}{r^2}\Big] && = 0 \\ & \implies \Big[\frac{M_f}{(M_f + M_s)^2} - \frac{\mu_S^2 M_f}{r^2}\Big] && = 0 \\ & \implies \frac{1}{(M_f + M_s)^2} = \frac{\mu_S^2}{r^2} \end{aligned}. \tag{32}$$

Similarly, the results indicate that $E \leq M_f$, which implies that $E = M_s + M_f$, implies that $F_f = 0$ since $M_f + F_f + M_s = E$. Thus, $(0, 0, \frac{r}{\mu_S})$ is the only invariant set in Ω satifying that $Q' = 0$ when $\mathscr{R}_0 = 1$. Thus, by LaSalle's invariance principle, the WFE is globally asymptotically stable in Ω when $\mathscr{R}_0 = 1$. Therefore, from (28) and (32) $\mathscr{R}_0 > 1$, $Q' > 0$ in the neighborhood of X_0. This makes X_0 unstable. For $\mathscr{R}_0 > 1$, the first term of (28) is positive hence $Q' > 0$. Therefore, we use continuity to show that Q' remains positive in a small neighborhood of X_0. □

When $\mathcal{R}_0 \leq 1$, the global asymptotic stability of the WFE rules out the existence of backward bifurcation. We use the uniform persistence results based on [13] and argument in proof of proposition 3.3 of [14] we can show that when $\mathcal{R}_0 > 1$ instability of X_0 implies that the system is uniformly persistent. This shows that uniform persistence and positive invariance of the compact set in Ω implies the existence of at least one positive equilibrium.

5. Numerical Simulations of the model system

5.1. Sensitivity analysis of parameters

We ascertain the contribution of each parameter in the model by carrying out the as sensitivity analysis in the control reproduction number $\mathcal{R}_0$ based on the values listed in Table 1.

5.1.1. Parameter Estimation

The parameter values are estimated and presented in Table 1. Our simulations are based on the these values.

Table 1. Estimates of parameter values

Parameter		Values	Source
β	the characteristic competition parameter averages over all the stages	0.357	Estimate
α	the primary sex ratio	0.5	Estimate
σ	the number of eggs laid per day for the E. Walker moth	4.725	Estimate
μ_{Mf}	mortality rate of fertile male	0.04	Estimate
μ_{Ff}	mortality rate for fertile female	0.03	Estimate
μ_s	mortality rate for sterile male	0.04	Estimate
r	Recruitment rate of sterile male	3.8424×10^3	Estimate

5.1.2. Elasticity indices

The elasticity index of a parameter p is given by $\frac{p\partial\mathcal{R}_0}{\mathcal{R}_0\partial p}$. Therefore, its a measure of the relative change in $\mathcal{R}_0$ to the relative change in p. A parameter with the largest elasticity magnitude has the greatest effect on $\mathcal{R}_0$ hence the spread of wild *E. walker* population. Table 2 show elasticity index calculated based on the parameter values in Table 1.

Table 2. Elasticity Indices based on parameters in $\mathcal{R}_0$

Parameter	Elasticity Index
α	1
β	1/2
μ_S	1
σ	1
K	-1/2
μ_F	-1/2
μ_M	-1/2
r	-1

The elasticity index is summarized in Figure 2

5.1.3. Numerical simulations for the model analysis

We present a numerical simulation for the system (1)-(3) based on parameter values in Table 1. The control reproduction number $\mathcal{R}_0$ in (18) is obtained as $\mathcal{R}_0 = 0.0021 < 1$. This implies that the wild population will be wiped out in the long-run. Figures 3 that the trajectories approach WFE $\left(M_f^*, F_f^*, M_s^*\right) \approx \left(0, 0, \left[\frac{r}{\mu_s} = 2500\right]\right)$.

Figure 3 indicate the population of wild (fertile) males M_f, wild (fertile) females F_f decreases while that of sterile male M_s increases over time for a period of 128 weeks before M_f reaches zero. The population of F_f stabilizes before reaching zero and remains constant after 120 weeks. The figure shows that this growth is slow, and would be detrimental to the sugarcane considering the maturity of sugarcane is around 18 months (78 weeks). This implies that by the time the M_s population stabilizes, the pest would have ravaged the sugarcane leaving farmers with no yields. Thus, a need for optimal control strategy to expedite the reduction of the population of M_f, and F_f is inevitable to reduce the time before stabilization of the population of the M_s.

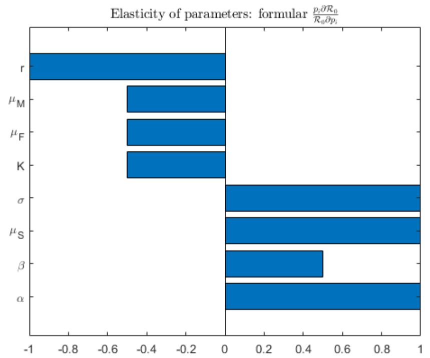

Fig. 2. Elasticity index of parameters in Table 2.

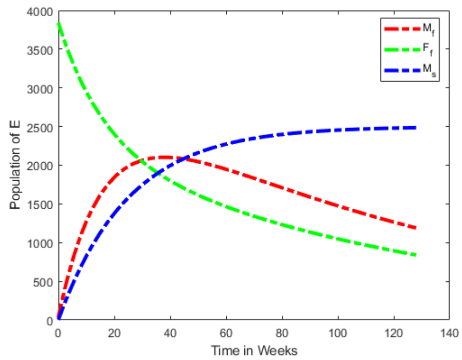

Fig. 3. The number of wild *E. walker* population in system (1) -(3) using parameter values in Table 1 with $M_{f_0} = 3.0739 * 10$, $F_{f_0} = 3.0739 * 10^3$, and $M_{s_0} = 0$ and $\mathcal{R}_0 = 0.0021$. The approximated equilibrium values are $\left(M_f^*, F_f^*, M_s^*\right) \approx (0, 0, 2500)$.

5.2. Sensitivity Analysis

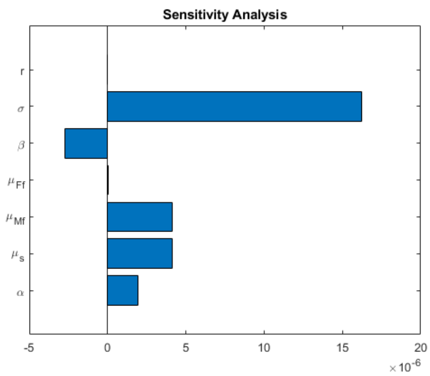

Fig. 4. Simulating the sensitivity of parameters $\alpha, \mu_s, \mu_{Mf}, \sigma, \beta, \mu_{Ff}, r$ with respect to $\mathcal{R}_C$. The graph indicate that σ (the number of eggs laid per day for the E. Walker moth) is the most sensitive parameter.

Figure 2 shows that σ, r, α is the most sensitive parameter in controlling the population of M_f, F_f, M_s. Therefore, a sensitivity analysis is set based on reducing σ.

6. Optimal Control applied to wild *E. walker* model

We extend the model (1)-(3) to reduce the number of eggs laid per day can reduce the population of wild *E. walker* population. No specific strategy in literature is recommended to wipe out the people of the *E. walker.* Thus, these are attempts that, speculatively, could reduce the population.

6.1. Introduction of the controls

From Table 2 and Figure 2 shows that σ, r, α has the greatest elasticity indices on $\mathscr{R}_0$. Thus can be easily be used to control the population of M_f, F_f, M_s. Therefore, we consider σ, r, α in reducing the population of wild *E. walker*.

6.1.1. Eggs laid

Let $g \in [0,1]$ be a time-dependent and Lebesgue measurable control representing a strategy to reduce the eggs laid per day, reducing the number of eggs hatched. The effect may include the timely release of the sterile males so that by the time the fertile males are mature to mate, the sterile males will have mated with the fertile female. The set of admissible eggs laid control is

$$G = \{g(t) : [0,T] \to [0,1] \text{ and } g \text{ is Lebesgue measurable}\}$$

6.1.2. Sex ratio

Let $s_r \in [0,1]$ be a time-dependent and Lebesgue measurable control representing a strategy to reduce the sex ratio, reducing the sex ratio between fertile male and female thus, increasing the number of sterile eggs laid. The effect may include the earlier release of the sterile males so that by the time the fertile males are mature to mate, the sterile males will have mated with the fertile female, most eggs laid are sterile. The set of admissible eggs laid control is

$$s_r = \{s_r(t) : [0,T] \to [0,1] \text{ and } s_r \text{ is Lebesgue measurable}\}$$

The controls can be denoted as $u = (g, s_r)$ and the set of admissible controls $U = G \times S_R$.

6.2. The extended mathematical model

The portion of the fertile males who are affected by the control are

$$\frac{dM_f}{dt} = (1-g)\sigma(1-(1-s_r)\alpha)\frac{F_f}{K} - \mu_M M_f \tag{33}$$

$$\frac{dF_f}{dt} = (1-g)\sigma(1-s_r)\alpha\frac{M_f}{M_f+M_s}\frac{\beta}{M_f+M_s} - \mu_F F_f \tag{34}$$

$$\frac{dM_s}{dt} = r - \mu_s M_S \tag{35}$$

With initial conditions

$$M_f(0) = M_{f_0}, F_f(0) = F_{f_0}, M_s(0) = M_{s_0}. \tag{36}$$

The success of intervention is measured based on its ability to reduce the population growth of wild population of *E. walker*. A possible cost include the cost of new or increased release of the sterile male. Therefore, the control $u = (g, s_r)$ is considered optimal if it minimizes the objective function defined as

$$J = \int_0^T \Big(A_1\Big[(1-g)\sigma(1-s_r)\alpha\frac{M_f}{M_f+M_s}\frac{\beta}{M_f+M_s}\Big] + A_2 g^2 + A_3 s_r^2\Big)dt \tag{37}$$

where A_1 is the cost of timely release of sterile males, $A_2 = A_3$ is the cost of implementing the timely release of sterile males per unit time. A_1, A_2 and A_3 are the balancing coefficients transforming the integrand into cost per time unit. $(A_1\Big[(1-g)\sigma(1-s_r)\alpha\frac{M_f}{M_f+M_s}\frac{\beta}{M_f+M_s}\Big]$ respresent the cost of timely release. We can thus state the optimal control as follows:

$$\min_{u\in U} J(u) \tag{38}$$

subject to (33)-(35) and initial conditions (36).

6.3. Existence of optimal control

Theorem 6.1.
There exists an optimal control u^* and the corresponding solution $\left(M_f^*, F_f^*, M_s^*\right)$ to the initial value problem given by (36) that minimizes the objective function given by (37) on U.

Proof. The initial value problem (33) -(35) can be written as

$$X^{'} = f(t, X, u) \tag{39}$$
$$\text{with } X(0) = X_0. \tag{40}$$

It is established that the existence of optimal control via results of Theorem 4.1 of [15] are met based on the following conditions

1. There exist $\mathscr{C}_1$ and $\mathscr{C}_2$ such that
   a $|f(t, X, u)| \le \mathscr{C}_1(1 + X_1)$ and
   b $|f(t, X_1, u) - f(t, X_2, u| \le \mathscr{C}_2|X_1 - X_2|$, for all $t \ge 0$, $X_1, X_2 \in \{(M_f, F_f, M_s) \in \mathbb{R}^3_+ | M_f + F_f + M_s = E\}$, and u U, where $U = \{u = (g, s_r) : O \le g, s_r \le 1\}$
2. The set of control and corresponding state variables are non-empty (increasing or decreasing functions).
3. The control set U is convex and closed, $f(t, X, u) = \eta(t, X) + \varpi(t, X)u$ and M is convex on U, where $M = \Big(A_1\Big[(1 - g)\sigma(1 - s_r)\alpha\frac{M_f}{M_f + M_s}\frac{\beta}{M_f + M_s}\Big] + A_2 g^2 + A_3 s_r^2\Big)$ is the integrand in (37).
4. There exist $\mathscr{C}_3 > 0$, $\mathscr{C}_4 > 1$, $\mathscr{C}_5 \ge 0$, such that
   $M(t, X, u) \ge \mathscr{C}_3|u|^{\mathscr{C}_4} - \mathscr{C}_5$

Since f is C^1, conditions 1(a) and 1(b) are implied by suitable bounds on partial derivatives of f and on $f(t, 0, 0)$. Since f is continuous and bounded on finite time interval, Theorem 9.2.1 in [16] guarantees that we have at least one local solution. The set $U = \{(g, s - r) : g \in [0, 1]$ and $s_r \in [0, 1]\}$ is closed. By deifinition, the set $Q = \{g : g \in [0, 1]$ is Lebesgue measurable is convex if $g_2, g_2 \in Q$ and $\varphi_1 \in [0, 1]$ implying that $\Big[(1 - \varphi_1)g_1 + \varphi_1 g_2\Big] \in Q$

$(1 - \varphi_1)g_1 + \varphi_1 g_2 \ge 0$ since $\varphi_1, g_1, g_2 \in [0, 1]$,

and

$$\begin{aligned}(1 - \varphi_1)g_1 + \varphi_1 g_2 &\ge 0(1 - \varphi_1) + \varphi_1 \text{ since } \varphi_1, g_1, g_2 \le 1\\ &= 1\end{aligned}$$

Therefore, $(1 - \varphi_1)g_1 + \varphi_1 g_2$ lies in Q implies that Q is convex, thus G is convex since according to [17] the Cartesian of convex set is convex; $U = G \times S_R$ is convex set. The function f is linear in each control variable g and s_r, thus it can be written as $f(t, X, u) = \eta(t, X) + \varphi(t, X)u$. M is convext on U since it is quadratic in u and the constant A_2 and A_3 are positive.

□

6.4. Characterization of the controls

We use Pontryagin's principle state in [18] in Pg. (84-86) to find the best possible control for the system (33)-(35). We define the Hamiltonian H as follows;

$$\begin{aligned}H(X, u, p) = \quad & p.(f(t, X, u) + L(t, X, u)\\ = \quad & p_1 f_1 + p_2 f_2 + p_3 f_3 + L\\ = \quad & p_1\Big[(1 - g)\sigma(1 - (1 - s_r)\alpha)\frac{F_f}{K} - \mu_M M_f\Big]\\ + \quad & p_2\Big[(1 - g)\sigma(1 - s_r)\alpha\frac{M_f}{M_f + M_s}\frac{\beta}{M_f + M_s} - \mu_F F_f\Big]\\ + \quad & p_3\Big[r - \mu_s M_S\Big]\\ + \quad & \Big(A_1\Big[(1 - g)\sigma(1 - s_r)\alpha\frac{M_f}{M_f + M_s}\frac{\beta}{M_f + M_s}\Big]\\ & + A_2 g^2 + A_3 s_r^2\Big)\end{aligned} \tag{41}$$

where $p = (p_1, p_2, P_3)$ and p_1, p_2, P_3 are adjoint variables for the state variables M_f, F_f, M_s.

Theorem 6.2.
Given an optimal solution (X^*, u^*) of the control problem (38), there exist p_1, p_2 and P_3, a solution set to the adjoint system

$$\begin{aligned} \dot{p}_1 = & -\frac{\partial H}{\partial M_f} &= \mu_M p_1 + \frac{\alpha\beta\sigma(g-1)(s_r-1)}{(M_f+M_s)^2}\Sigma_1 \\ \dot{p}_2 = & -\frac{\partial H}{\partial F_f} &= \mu_F p_2 + \frac{\beta p_1\sigma(g-1)(\alpha(s_r-1)+1)}{K} \\ \dot{p}_3 = & -\frac{\partial H}{\partial M_s} &= \mu_S p_3 + \Sigma_2\Big[A_1 + p_2\Big] \end{aligned} \tag{42}$$

where $\Sigma_1 = \Big[\frac{2A_1 M_f}{M_f+M_s} + \frac{p_2(M_f-M_s)}{M_f+M_s} - A_1\Big]$,
$\Sigma_2 = \frac{2M_f\alpha\beta\sigma(g-1)(s_r-1)}{(M_f+M_s)^3}$
with transversality condition

$p_1(T) = 0,\ p_2(T) = 0,\ p_3(T) = 0$ *such that* $u^* = \min_{u\in U} H(X, p, u),\ t \in [0, T]$. *Furthermore, the controls can be characterised as*

$$g^* = \min\left(1, \max\left(0, \frac{\beta\sigma}{2A_2}\Big[\frac{M_f(1-s_r)}{(M_f+M_s)^2}\Big(A_1+\alpha p_2\Big)+\Sigma_3\Big]\right)\right)$$

where $\Sigma_3 = \frac{F_f p_1(\alpha(s_r-1)+1)}{K}$
and

$$s_r^* = \min\left(1, \max\left(0, \frac{\alpha\beta\sigma(g-1)\Sigma_4}{2A_3K(M_f+M_s)^2}\right)\right)$$

where
$\Sigma_4 = (F_f M_f^2 p_1 - KM_f p_2 - A_1 KM_f + F_f M_s^2 p_1 + 2F_f M_f M_s p_1)$

Proof. The optimal control is derived from the optimality condition $\frac{\partial H}{\partial u}|_{u^*} = 0$.

$$\begin{aligned} \frac{\partial H}{\partial g}\Big|_{g^*} &= 0 \\ &\implies g^* = \frac{\beta\sigma}{2A_2}\Big[\frac{M_f(1-s_r)}{(M_f+M_s)^2}\Big(A_1+\alpha p_2\Big)+\Sigma_3\Big] \end{aligned} \tag{43}$$

$$\begin{aligned} \frac{\partial H}{\partial s_r}\Big|_{s_r*} &= 0 \\ &\implies s_r^* = \frac{\alpha\beta\sigma(g-1)\Sigma_4}{2A_3K(M_f+M_s)^2} \end{aligned} \tag{44}$$

We consider the properties of the optimal control space to get;

$$g^* = \begin{cases} 0, & \text{if } \frac{\beta\sigma}{2A_2}\Big[\Pi_\Sigma+\Sigma_3\Big] \le 0, \\ \frac{\beta\sigma}{2A_2}\Big[\Pi_\Sigma+\Sigma_3\Big], & \text{if } 0 < \frac{\beta\sigma}{2A_2}\Big[\Pi_\Sigma+\Sigma_3\Big] < 1, \\ 1, & \text{if } \frac{\beta\sigma}{2A_2}\Big[\Pi_\Sigma+\Sigma_3\Big] \ge 1. \end{cases} \tag{45}$$

where $\Pi_\Sigma = \frac{M_f(1-s_r)}{(M_f+M_s)^2}\Big(A_1+\alpha p_2\Big)$. Hence g^* can be characterized as
$g^* = \min\Big(1, \max\Big(0, \frac{\beta\sigma}{2A_2}\Big[\Pi_\Sigma+\Sigma_3\Big]\Big)\Big)$.
s_r^* can also be characterized as
$s_r^* = \min\Big(1, \max\Big(0, \frac{\alpha\beta\sigma(g-1)\Sigma_4}{2A_3K(M_f+M_s)^2}\Big)\Big)$.

Additionally, we note from (43) and (44) that $\frac{\partial^2 H}{\partial g^2}\Big|g^* = 2A_2 > 0$ and $\frac{\partial^2 H}{\partial s_r^2}\Big|s_r^* = 2A_3 > 0$. therefore, A_2 and A_3 are positive constants introduced in (37), which indicate that s $u^* = \big(g^*, s_r^*\big)$ minimizes the Hamiltonian function $H(X, p, u)$. □

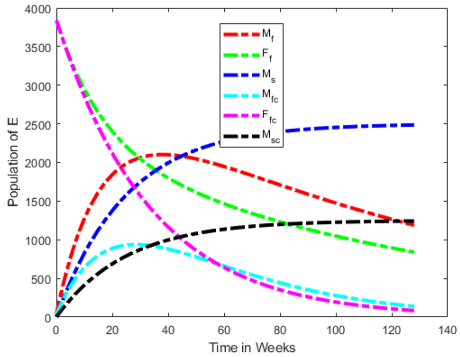

Fig. 5. Simulating the population of M_f and M_{fc} for the entire period understudy before and after optimal control strategy, respectively.

7. Numerical Simulation for the control

A simulation of the extended (33) -(35) to show the effects of the optimal control strategy is presented in the Figures 5 based on the parameter values given in Table 1. A comparison is made of the population of the *E. walker* in each compartment with and without optimal control.

In Figure 5 a decrease in fertile the eggs and sex ratio by 60% reduces the population of M_f, F_f and M_s by an equivalent proportion. A comparison between M_f, F_f and M_s in the presence of control $|_c$ shows that the trajectories for M_f, and F_f steeply falls towards zero faster compared to in the absence of optimal control. The trajectories take shorter time to reach zero in the presence of optimal control compared to in the absence.

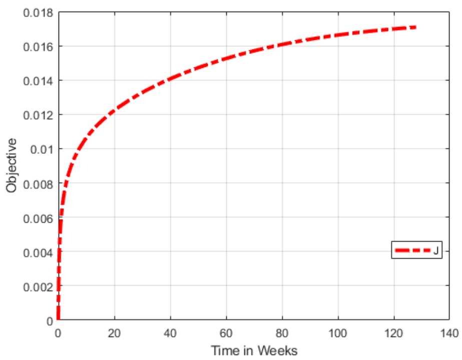

Fig. 6. Simulation of weekly cost of timely release in order to reduce the fertile eggs and sex ratio.

Figure 6 indicates that between week 1 and week 10, the cost increases by 1% and then gradually curves towards week 40. Between week 1o and week 40, the cost increases by 0.4%. This indicates that the cost of implementing the control is very small compared to the damage the *E. walker* pest can bring to sugarcane farmers.

Figure 7 indicates that the intervention reduces the egg population throughout the study period, explaining why there is a sharp drop in the population of fertile males and females in Figure 5 during the control.

Similarly, Figure 8 indicates that the intervention reduces the sex ratio between the fertile male and female throughout the study period, explaining why there is a sharp drop in the population of fertile males and females in Figure 5 during the control.

Figures 6-8 explains the effectiveness of the control and the cost of implementing the control. The figures suggest that the cost is very small; thus, such interventions could be feasible in real life.

8. Conclusion

The study proposes a novel model for formulating the dynamics of the E. walker population with SIT. The model proposed consists of four compartments, with only three active compartments. The proposed governing equations were found to be feasible for application in real-life; that is, the solutions to the equations exist and unique and positive. The model equations had only wild free equilibrium (WFE) points.

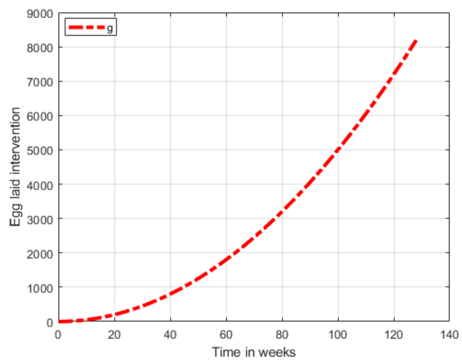

Fig. 7. Numerical simulating the effort of reducing fertile eggs.

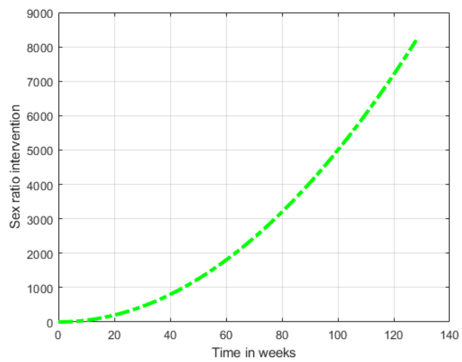

Fig. 8. Numerical simulating the effort of reducing sex ratio of fertile male and female.

The basic reproductive number $\mathscr{R}_0$ was obtained to be less than 1. This indicated that the wild E. walker population under SIT will wipe out after some time. However, the question lingered about the duration before it is wiped out. The local stability of the WFE indicated that the established $\mathscr{R}_0$ was locally asymptotically stable and $\mathscr{R}_0 < 1$. The global stability of WFE via the matrix theoretic method showed that WFE is globally asymptotically stable when $\mathscr{R}_0 < 1$.

The numerical simulation answered the question of the duration before the wild population is wiped out. The simulation revealed that the wild E. walker population under SIT will be wiped out after more than 200 weeks. This is unrealistic, considering that the sugarcane matures after approximately 78 weeks. This indicated a need for an optimal control strategy to expedite the 'wipe-out'. Elasticity analysis of the model parameters based on $\mathscr{R}_0$ indicated that a possible control lies in controlling the eggs laid and sex ratio. An extended mathematical control based on eggs laid and sex ratio existed, and the approach can minimize its Hamiltonian function.

The numerical simulation showed that the population of the wild E. walker is wiped out after approximately 130 weeks, an indication of the effect of the control. However, this still needs to be satisfactory, necessitating future studies and a sign of research value into the concept. A simulation of the objective cost function indicated an increase in the cost within the first few weeks, followed by an almost flat curve. This indicates that the cost of implementing the control only increases at the onset of the application and then flattens. This shows that the proposed control is achievable and will support the farmers' economics. A simulation of the efforts to reduce the eggs and increases the sex ratio between sterile male and fertile female was also effective due to the growth of both curves. Future studies into the area need to refocus on the timelines to investigate other strategies to reduce the wild E. walker population below the sugarcane maturity stage.

Acknowledgement

The work was supported by Department of Pure and Applied Mathematics, Jomo Kenyatta University of Agriculture and Technology (JKUAT), Nairobi, Kenya.

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