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Remarks on Essential Maximal Numerical Range of Aluthge Transform

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Abstract. This paper focuses on the properties of the essential maximal numerical range of Aluthge transform $\widetilde{T}$. For instance, among other results, we show that the essential maximal numerical range of Aluthge transform is nonempty and convex. Further, we prove that the essential maximal numerical range of Aluthge transform $\widetilde{T}$ is contained in the essential maximal numerical range of T. This study is therefore an extention of the research on Aluthge transform which was begun by Aluthge in his study of $p-$hyponormal operators.

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Introduction

Let $B(X)$ denote the algebra of bounded linear operators acting on a complex Hilbert space X. Let us recall that the Aluthge transform $\widetilde{T}$ of T is the operator $|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. Here, we denote by T the bounded linear operator on a complex Hilbert space X and let $T = U|T|$ be any polar decomposition of T with U a partial isometry and $|T| = (T^*T)^{\frac{1}{2}}$. Recall also that if $\ker T$ is the kernel of a bounded linear operator T then a bounded linear operator $T \in B(X)$ is said to be an isometry if $\|Tx\| = \|x\|\ \forall x \in X$. We say that T is a partial isometry if it is an isometry on the orthogonal complement of its kernel, that is, for every $x \in \ker(T)^{\perp}$, $\|Tx\| = \|x\|$.

After its conception in 1900 by Aluthge [1], the notion of Aluthge transform and the study of its properties with their generalizations has attracted the attention of many authors such as in [5], [6] among others. This extensive research is because Aluthge transform is a very useful tool for studying some operator classes. Especially, it is used by many researchers in the study of $p-$hyponormal and semi-hyponormal operators. In 2007, Guoxing Ji, Ni Liu and Ze Li [6] together showed that the essential numerical range of Aluthge transform is contained in the essential numerical range of T. It is also known that spectrum of the normal operator T coincides with the spectrum of Aluthge transform $\widetilde{T}$, that is, $\sigma(T) = \sigma(\widetilde{T})$. See [5] for this and more. Aluthge transform $\widetilde{T}$ of an $m-$tuple operator $T = (T_1, ..., T_m) \in B(X)$ was studied in [2], [3] and [4] and interesting results established.

Essential Maximal Numerical Range of Aluthge Transform

This section establishes some of the properties of the essential maximal numerical range of Aluthge transform. If $T = U|T|$ is any polar decomposition of an operator $T \in B(X)$ with U a partial isometry and $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ then we denote the essential maximal numerical range of Aluthge transform as $\mathrm{Max}W_e(\widetilde{T})$ and define it as

$$\mathrm{Max}W_e(\widetilde{T}) = \{r \in \mathbb{C} : \langle \widetilde{T}x_n, x_n\rangle \to r, x_n \to 0 \text{ weakly and } \|\widetilde{T}x_n\| \to \|\widetilde{T}\|_e\}.$$

Theorem 1. *Let $T = U|T|$ be any polar decomposition of an operator $T \in B(X)$. If $r \in \mathrm{Max}W_e(\widetilde{T})$ for any $r \in \mathbb{C}$, then there exists an orthonormal sequence $\{x_n\} \in X$ such that*

$\langle \widetilde{T}x_n, x_n \rangle \to r$ *and* $\|\widetilde{T}x_n\| \to \|\widetilde{T}\|_e$.

Proof. Suppose $r \in \text{Max}W_e(\widetilde{T})$. Then there is a sequence $\{x_n\}$ of vectors such that

$\langle \widetilde{T}x_n, x_n \rangle \to r$, $\|x_n\| = 1$, $x_n \to 0$ weakly and $\|\widetilde{T}x_n\| \to \|\widetilde{T}\|_e$.

Choosing the set $\{x_1, ..., x_n\}$ which satisfy $|\langle \widetilde{T}x_n, x_n \rangle - r| < \frac{1}{i}$ $\forall i$ and letting $\mathcal{M}$ be the subspace spanned by $x_1, ..., x_n$ and P be the projection onto $\mathcal{M}$ then we have $\|Px_n\| \to 0$ as $n \to \infty$. Let $z_n = \|(I-P)x_n\|^{-1}((I-P)x_n)$. We obtain $\widetilde{T}z_n = \|(I-P)x_n\|^{-1}(\widetilde{T}(I-P)x_n)$. This gives

$$\begin{aligned}\langle \widetilde{T}z_n, z_n \rangle &= \langle \|(I-P)x_n\|^{-1}(\widetilde{T}(I-P)x_n), \|(I-P)x_n\|^{-1}(\widetilde{T}(I-P)x_n) \rangle \\ &= \|(I-P)x_n\|^{-2}\{\langle \widetilde{T}x_n, x_n \rangle - \langle \widetilde{T}x_n, Px_n \rangle - \langle \widetilde{T}Px_n, x_n \rangle + \langle \widetilde{T}Px_n, Px_n \rangle\} \to r \\ &\text{as } n \to \infty.\end{aligned}$$

We then choose n large enough such that $|\langle \widetilde{T}z_n, z_n \rangle - r| < \frac{1}{n+1}$. If we let $z_n = x_{n+1}$ we get $|\langle \widetilde{T}x_{n+1}, x_{n+1} \rangle - r| < \frac{1}{n+1}$ which completes the proof.

Lemma 2. *Suppose* $T \in B(X)$, $\|T\| = 1$, $\|x_n\| = 1$ *and* $T = U|T|$ *any polar decomposition of an operator* $T \in B(X)$. *If* $\|\widetilde{T}x_n\|^2 \geq (1-\epsilon)$, *then* $\|\left(\widetilde{T}^*\widetilde{T} - I\right)x_n\|^2 \leq 2\epsilon$.

Proof. Since $\widetilde{T}^*\widetilde{T} - I \geq 0$ it follows that,

$$\begin{aligned}\|\left(\widetilde{T}^*\widetilde{T} - I\right)x_n\|^2 &= \|\widetilde{T}^*\widetilde{T}x_n\|^2 - 2\|\widetilde{T}x_n\|^2 + \|x_n\|^2 \\ &\leq 2\left(1 - \|\widetilde{T}x_n\|^2\right) \\ &\leq 2\epsilon.\end{aligned}$$

Theorem 3. *Let* $T = U|T|$ *be any polar decomposition of an operator* $T \in B(X)$. *Suppose that for a point* $r \in \mathbb{C}$ *there exists an orthonormal sequence* $\{x_n\} \in X$ *such that*

$\langle \widetilde{T}x_n, x_n \rangle \to r$ *and* $\|\widetilde{T}x_n\| \to \|\widetilde{T}\|_e$. *Then* $r \in \text{Max}W_e(\widetilde{T})$

Proof. Assume without loss of generality that for a point $r \in \mathbb{C}$ there exists an orthonormal sequence $\{x_n\} \in X$ such that $\langle \widetilde{T}x_n, x_n \rangle \to r$ and $\|\widetilde{T}x_n\| \to \|\widetilde{T}\|_e$. Since $\|x_n\| = 1$ and every orthonormal sequence $\{x_n\}$ converges weakly to zero, it implies that $r \in \text{Max}W_e(\widetilde{T})$.

Theorem 4. *The set* $\text{Max}W_e(\widetilde{T})$ *is nonempty and convex.*

Proof. We prove that $\text{Max}W_e(\widetilde{T})$ is nonempty. To do this, from Theorem 1, there exists an orthonormal sequence $\{x_n\} \in X$ such that $\langle \widetilde{T}x_n, x_n \rangle \to r$ and $\|\widetilde{T}x_n\| \to \|\widetilde{T}\|_e$. Thus the sequence $\{\langle \widetilde{T}x_n, x_n \rangle\}$ is bounded. Choose a subsequence and assume that $\langle \widetilde{T}x_n, x_n \rangle$ converges. Then $\text{Max}W_e(\widetilde{T})$ is nonempty

To show convexity, let $r, \mu \in \text{Max}W_e(\widetilde{T})$. Since $r, \mu \in \text{Max}W_e(\widetilde{T})$, it implies that there exist orthonormal sequences $x_n, y_n \in X$ such that $\|x_n\| = 1 = \|y_n\|$, $\langle \widetilde{T}x_n, x_n \rangle \to r$ and $\|\widetilde{T}x_n\| \to \|\widetilde{T}\|_e$. Also $\langle \widetilde{T}y_n, y_n \rangle \to \mu$ and $\|\widetilde{T}y_n\| \to \|\widetilde{T}\|_e$. Let $\mathcal{M}_n$ be a subspace spanned by x_n and y_n and $\mathcal{P}_n$ be a projection of X onto M_n. Suppose $\widetilde{T}_n = \mathcal{P}_n\widetilde{T}\mathcal{P}_n$, then $\langle \widetilde{T}x_n, x_n \rangle = \langle \widetilde{T}y_n, y_n \rangle$ are in the numerical range of $\mathcal{P}_n\widetilde{T}\mathcal{P}_n$. By Toeplitz-Hausdorff Theorem, $W(\mathcal{P}_n\widetilde{T}\mathcal{P}_n)$ is convex and so for each n we can choose α_n, β_n with $\nu_n = \alpha_n x_n + \beta_n y_n = 1$ (where ν_n is a sequence in X). If η is a point on the line segment joining r and μ then $\langle \widetilde{T}\nu_n, \nu_n \rangle \to \eta$ and $\|\nu_n\|l = 1$. Note that $|\langle x_n, y_n \rangle| \leq \theta < 1$ for n sufficiently large. This implies that the angle between x_n and y_n is bounded away from 0. Therefore, there exists a constant M such that $|\alpha_n| \leq M$ and $|\beta_n| \leq M$ for n sufficiently large, where $\|\alpha_n x_n + \beta_n y_n\| = 1$. By Lemma 2, $\|\widetilde{T}\nu_n\| = \langle \widetilde{T}^*\widetilde{T}\nu_n, \nu_n \rangle = \|\nu_n\|^2 - 2M\epsilon$ where $\epsilon \to 0$. That is, $\|\left(\widetilde{T}^*\widetilde{T} - I\right)x_n\| \to 0$ and $\|\left(\widetilde{T}^*\widetilde{T} - I\right)y_n\| \to 0$ as $n \to \infty$. Thus $\|\widetilde{T}\nu_n\| \to 1$ as $n \to \infty$ implying that $\|\widetilde{T}\nu_n\| \to \|\widetilde{T}\|$ as $n \to \infty$.

Theorem 5. *Let $T = U|T|$ be any polar decomposition of an operator $T \in B(X)$. Suppose that for a point $r \in \mathbb{C}$ there exists an orthonormal sequence $\{x_n\} \in X$ such that $\langle \widetilde{T}x_n, x_n\rangle \to r$ and $\|\widetilde{T}x_n\| \to \|\widetilde{T}\|_e$. Then there exists an infinite - dimensional projection P such that*

$P(\widetilde{T} - rI)P \in \mathcal{K}(X)$ and $\|\widetilde{T}P\|_e = \|\widetilde{T}\|_e$.

Proof. Let $\{x_n\} \in X$ be an orthonormal sequence such that $\langle \widetilde{T}x_n, x_n\rangle \to r$ and $\|\widetilde{T}x_n\| \to \|\widetilde{T}\|_e$. By passing to a subsequence we can assume that

$$\sum_{n=1}^{\infty} |\langle (\widetilde{T} - r)x_n, x_n\rangle|^2 < \infty \tag{1}$$

Let $n_1 = 1$. Then $\sum_{n=1}^{\infty} |\langle (\widetilde{T} - r)x_{n_1}, x_n\rangle|^2 \leq \|(\widetilde{T} - r)x_{n_1}\|^2$ and $\sum_{n=1}^{\infty} |\langle (\widetilde{T} - r)x_n, x_{n_1}\rangle|^2 \leq \|(\widetilde{T} - r)^* x_{n_1}\|^2$. Thus, by Bessel's inequality, there is an integer $n_2 > n_1$ such that $\sum_{n=n_2}^{\infty} |\langle (\widetilde{T} - r)x_{n_1}, x_n\rangle|^2 < 2^{-1}$ and $\sum_{n=n_2}^{\infty} |\langle (\widetilde{T} - r)x_n, x_{n_1}\rangle|^2 < 2^{-1}$. If this procedure is repeated, a strictly increasing sequence $\{n_t\}_{t=1}^{\infty}$ of positive integers is obtained such that we have

$$\sum_{n=n_{t+1}}^{\infty} |\langle (\widetilde{T} - r)x_{n_t}, x_n\rangle|^2 < 2^{-t}$$

and

$$\sum_{n=n_{t+1}}^{\infty} |\langle (\widetilde{T} - r)x_n, x_{n_t}\rangle|^2 < 2^{-t} \tag{2}$$

(1) and (2) both imply that

$$\sum_{t,l=1}^{\infty} |\langle (\widetilde{T} - r)x_t, x_{n_l}\rangle|^2 < \infty \tag{3}$$

If P is an orthogonal projection onto the subspace $\mathcal{M}$ spanned by $x_{n_1}, x_{n_2}, \ldots$, then

$$\sum_{t,l=1}^{\infty} |\langle (P\widetilde{T}P - rP)x_{n_t}, x_{n_l}\rangle|^2 = \sum_{t,l=1}^{\infty} |\langle (\widetilde{T} - r)x_{n_t}, x_{n_l}\rangle|^2 < \infty \text{ by (3)},$$

hence PTP is a Hilbert - Schmidt operator and therefore $P\widetilde{T}P - rP \in \mathcal{K}(X)$.

Remark 6. We remark the following:

(i) An equivalent definition of $\mathrm{Max}W_e(\widetilde{T})$ can be formulated when the orthonormal sequences are replaced by the weakly convergent sequence $\{x_n\}$ as shown in the theorem below.

(ii) $\mathrm{Max}W_e(\widetilde{T}) = \mathrm{Max}W_e(\widetilde{T + K})$ if K is a compact operator.

Theorem 7. *Let $T = U|T|$ be any polar decomposition of an operator $T \in B(X)$. Suppose that for a point $r \in \mathbb{C}$ there exists a sequence $\{x_n\} \in X$ of vectors converging weakly to $0 \in X$ such that*

$\langle \widetilde{T}x_n, x_n\rangle \to r$ and $\|\widetilde{T}x_n\| \to \|\widetilde{T}\|_e$. Then $r \in \mathrm{Max}W_e(\widetilde{T})$.

Proof. Suppose that for a point $r \in \mathbb{C}$ there is a sequence $\{x_n\} \in X$ such that $\langle \widetilde{T}x_n, x_n\rangle \to r$ and $\|\widetilde{T}x_n\| \to \|\widetilde{T}\|_e$. Since every sequence $\{x_n\} \to 0$ weakly, and $\|x_n\| = 1,$ we have $r \to \text{Max}W_e(\widetilde{T})$.

Theorem 8. *Suppose $T = U|T|$ is any polar decomposition of an operator $T \in B(X)$ and $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. Then $r \in \text{Max}W_e(\widetilde{T})$ if there exists an infinite-dimensional projection P such that $P(\widetilde{T} - rI)P \in \mathcal{K}(X)$ and $\|\widetilde{T}P\|_e = \|\widetilde{T}\|_e$.*

Proof. Let $P \in B(X)$ be an infinite dimensional projection such that $P\widetilde{T}P \in \mathcal{K}(X)$ and $\|\widetilde{T}P\|_e = \|\widetilde{T}\|_e$. Then there is an orthonormal sequence $\{x_n\} \in X$ such that $Px_n = x_n \forall n$ and $\|\widetilde{T}Px_n\| \to \|\widetilde{T}P\|_e$. Since $\|\widetilde{T}P\|_e = \|\widetilde{T}\|_e$ we get $\|\widetilde{T}x_n\|_e \to \|\widetilde{T}\|_e$. Let $K \in \mathcal{K}(X)$. Since $P\widetilde{T}P = K + rP$ implies $\langle (P\widetilde{T}P - rP)x_n, x_n\rangle = \langle Kx_n, x_n\rangle$ then $\langle \widetilde{T}x_n, x_n\rangle = r + \langle Kx_n, x_n\rangle$. From the orthonormality of sequence $\{x_n\}$, we get Kx_n converging weakly to 0 in norm as $n \to \infty$. Therefore, $\langle \widetilde{T}x_n, x_n\rangle \longrightarrow r$ as $n \to \infty$ implying $r \in \text{Max } W_e(\widetilde{T})$.

The above results relating to the essential maximal numerical range of Aluthge transform can be summed up as shown below.

Corollary 9. *Let $T = U|T|$ be any polar decomposition of an operator $T \in B(X)$. Each of the following conditions is necessary and sufficient in order that $r \in \text{Max}W_e(\widetilde{T})$.*

(i) $\langle \widetilde{T}x_n, x_n\rangle \to r$ *and* $\|\widetilde{T}x_n\| \to \|\widetilde{T}\|_e$ *for some orthonormal sequence* $\{x_n\}$.

(ii) $\langle \widetilde{T}x_n, x_n\rangle \to r$ *and* $\|\widetilde{T}x_n\| \to \|\widetilde{T}\|_e$ *for some sequence* $\{x_n\} \in X$ *of vectors converging weakly to* $0 \in X$.

(iii) $P(\widetilde{T} - rI)P \in \mathcal{K}(X)$ *and* $\|\widetilde{T}P\|_e = \|\widetilde{T}\|_e$ *for an infinite - dimensional projection* P.

We end this section by examining the relationship between $\text{Max}W_e(T)$ and $\text{Max}W_e(\widetilde{T})$ using the following theorem.

Theorem 10. *Let $T \in B(X)$ and $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. Then* $\text{Max}W_e(\widetilde{T}) \subseteq \text{Max}W_e(T)$.

Proof. Assume, without loss of generality that $\|\widetilde{T}\| = \|T\| = 1$ and let $r \in \text{Max}W_e(\widetilde{T})$. Then, there exists a sequence $\{x_n\} \in X$ of unit vectors converging weakly to $0 \in X$ such that

$\langle \widetilde{T}x_n, x_n\rangle \to r$ and $\|\widetilde{T}x_n\| \to \|\widetilde{T}\|_e$.

Then, $\||T|^{1/2}x_n\| = \||T|^{1/2}\| = 1$ as $n \to \infty$ and $\|(1 - |T|)x_n\| = 0$ as $n \to \infty$.
Also, $\|(1 - |T|^3)x_n\| = 0$ as $n \to \infty$. Thus, $\lim\limits_{n\to\infty} \|T|T|^{1/2}x_n\| = \lim\limits_{n\to\infty} \langle |T|^3x_n, x_n\rangle = 1 = \|T\|$. And,

$$\begin{aligned}
\lim_{n\to\infty} |\langle \widetilde{T}x_n, x_n\rangle - \langle T\sqrt{|T|}x_n, \sqrt{|T|}x_n\rangle| &= \lim_{n\to\infty} |\langle U\sqrt{|T|}x_n, \sqrt{|T|}x_n\rangle - \langle T\sqrt{|T|}x_n, \sqrt{|T|}x_n\rangle| \\
&= \lim_{n\to\infty} |\langle (U|T|^{1/2} - U|T||T|^{1/2})x_n, |T|^{1/2}x_n\rangle| \\
&= \lim_{n\to\infty} |\langle (U|T|^{1/2})(1 - |T|)x_n, |T|^{1/2}x_n\rangle| \\
&\leq \lim_{n\to\infty} \|U|T|^{1/2}\|\|(1 - |T|)x_n\|\||T|^{1/2}x_n\| \\
&= 0.
\end{aligned}$$

If we let $z_n = (|T|^{1/2}x_n)/(\||T|^{1/2}x_n\|)$ then $\{z_n\} \in X$ is a sequence of unit vectors converging weakly to $0 \in X$ such that $\langle Tz_n, z_n\rangle \to r$ and $\|Tz_n\| \to \|T\|_e$. Thus $r \in \text{Max}W_e(T)$.
Hence $\text{Max}W_e(\widetilde{T}) \subseteq \text{Max}W_e(T)$.

Conclusion

This paper established some of the properties of the essential maximal numerical range of Aluthge transform $\widetilde{T}$. Among other results, the study proved that the essential maximal numerical range of Aluthge transform $\widetilde{T}$ is contained in the essential maximal numerical range of the operator T.

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