ON THE JOINT ESSENTIAL MAXIMAL NUMERICAL RANGES

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Abstract

The concept of maximal numerical range of a bounded operator T on $B(X)$ was introduced and studied in by Stampfli who used it to derive an identity for the norm of derivation. This concept was later generalised by Ghan to the Joint maximal numerical range, $MaxW_m(T)$, of an m-tuple of operator $T=(T_1,\dots,T_m)\in B(X)$. In 1997, Fong introduced the essential maximal numerical range to study the norm of a derivation on Calkin algebra. The Joint essential maximal numerical range was studied by Khan and certain results analogous to the single operator case proved. Khan also illustrated that the joint essential maximal numerical range can be empty. In the present paper, we show the equivalent definitions of the joint essential maximal numerical range $MaxW_{em}(T)$ and also show that the Joint essential maximal numerical range is nonempty, compact and convex. We also show that each element in the joint essential maximal numerical range is a star center of the joint maximal numerical range.

1. INTRODUCTION

Denote by X a complex Hilbert space with inner product $\langle .,. \rangle$ and norm $\|\cdot\|$ and denote by $B(X)$ the algebra of (bounded) linear operators acting on X.

The Joint Maximal Numerical range of $T=(T_1,\dots,T_m)\in S(X)^m$, denoted by $MaxW_m(T)$, is defined as,

$$MaxW_m(T)=\{r\in\mathbb{C}^m:\langle T_k x_n,x_n\rangle\to r_k,$$
$$where\ x_n\in X;\ \|x_n\|=1\ and\ \|T_k x_n\|\to\|T_k\|;1\le k\le m\}.$$

In the case k = 1, it is the usual maximal numerical range of an operator T denoted by $MaxW(T)$ and defined as

$$MaxW(T)=\{r\in\mathbb{C}^m:\langle T_k x_n,x_n\rangle\to r_k, where\ x_n\in X;\ \|x_n\|=1\ and\ \|Tx_n\|\to\|T\|\}.$$

From the properties of the joint Maximal numerical range, we note that $MaxW_m(T)$ does not have translation property by scalar, that is

$$MaxW_m(\beta T+\alpha I)\neq MaxW_m(T)+\propto\ \forall\ \beta,\alpha\in\mathbb{C}^m.$$

In particular, it is known that $MaxW_m(T)\cap\ MaxW_m(T+\beta)=\emptyset$ for any $0\neq\beta=(\beta_1,\dots,\beta_m)\in\mathbb{C}^m$ (see [3]).

The Joint essential maximal numerical range, denoted by $MaxW_m(T)$, is defined as

$MaxW_m(T) = \{r \in \mathbb{C}^m : \langle T_k x_n, x_n \rangle \to r_k, x_n \to 0 \ \ weakly\ and\ \|T_k x_n\| \to \|T_k\|_e \ 1 \le k \le m\}$

Here, $\|T_k\|_e$ denotes the essential norm of T defined by

$\|T_k\|_e = \inf\{\|T + K\| : K \in \mathcal{K}(X)\}$

Where $\mathcal{K}(X)$ is the ideal of all compact operators in $B(X)$. In [2], it was shown that $MaxW_{em}(T) \cap MaxW_{em}(T+\beta) = \emptyset$ for $0 \neq \beta = (\beta_1, \dots, \beta_m) \in \mathbb{C}^m$. In the case k = 1, the Joint essential maximal numerical range becomes the usual essential maximal numerical range, $MaxW_{em}(T)$ defined as

$MaxW_m(T) = \{r \in \mathbb{C}^m : \langle T_k x_n, x_n \rangle \to r_k, x_n \to 0 \ \ weakly\ and\ \|T x_n\| \to \|T\|_e \}.$

Let $\mathcal{A}$ be a complex normed algebra with unit e and let $T = (T_1, \dots, T_m) \in \mathcal{A}^m$. The Joint (algebra) maximal numerical range, denoted by $MaxV_m(T, \mathcal{A})$, of an element

$T = (T_1, \dots, T_m) \in \mathcal{A} \in A$ is defined by

$MaxV_m(T, \mathcal{A}) = \{f(T_1, \dots, f(T_m) : f \in \mathcal{A}^*)\}$ where $\mathcal{A}^*$ is the set of all maximal states of T.

Recall that a linear functional f on $\mathcal{A}$ is a state if $\|f\| = 1 f(e)$ and that if $f(T^*T) = \|T\|^2$ then the state f is maximal for T. We shall, if there is no confusion, abbreviate $MaxV_m(T, \mathcal{A})$ as $MaxV_m(T)$In this paper, we show the equivalent definitions of the joint essential maximal numerical range $MaxW_{em}(T)$ and also show that the Joint essential maximal numerical range is nonempty, compact and convex. We also show that each element in the joint essential maximal numerical range is a star center of the joint maximal numerical range.

We remind the reader that the numerical range of $T \in B(X)$ is defined as $W(T) = \{\langle Tx, x \rangle : x \in X, \langle x, x \rangle = 1\}$ while the essential numerical range of $T \in B(X)$ is defined as $W_e(T) = \cap \{\overline{W(T+K)} : K \in \mathcal{K}(X)\}$.

2. MAIN RESULTS

Lemma 1.1. $MaxW(T)$ *is nonempty, closed and convex subset of the closure of numerical range.*

The Proof of the lemma can be found in Stampfli [4]

Lemma 1.2. $MaxW_e(T)$ *is nonempty, closed and convex subset of the essential numerical range.*

See Fong [1] for the proof.

Theorem 1.3. *Let X be an infinite dimensional complex Hilbert space and* $T = (T_1, \dots, T_m) \in B(X)$. *Let* $r = (r_1, \dots, r_m) \in \mathbb{C}^m$ *.The following properties are equivalent:*

1.$r \in MaxW_{em}(T)$

2. There exists an orthonormal sequence $\{x_n\}_{n=1}^{\infty}$ *such that..*

$\langle T_k x_n, x_n \rangle \to r_k$ and $\|T_k x_n\| \to \|T_k\|_e$;$1 \le k \le m.$

3. There exists a sequence $\{x_n\}_{n=1}^{\infty} \in X$ *of vectors converging weakly to* 0 $\in X$ *such that* $\langle T_k x_n, x_n \rangle \to r_k$ and $\|T_k x_n\| \to \|T_k\|_e$;$1 \le k \le m.$

4. There exists an infinite - dimensional projection P such that $P(T_k - r_k I)P \in \mathcal{K}(X)$ and $\|T_k P\|_e = \|T_k\|_e$; $k = 1, \dots, m.$

Proof. 1 ⇔ 2 and 1 ⇔ 4 was proved by Khan [2].

To prove 2 ⇔ 4*;*

Let $\{x_n\}_{n=1}^{\infty} \in X$ be an orthonormal sequence such that

$\langle T_k x_n, x_n \rangle \to r_k$ and $\|T_k x_n\| \to \|T_k\|_e$;$1 \le k \le m.$

By passing to a subsequence we can assume that

$\sum_{n=1}^{\infty}|\langle(T_k - r_k)x_n, x_n\rangle|^2 < \infty$ (*)

Let $n_1 = 1$. Then

$$\sum_{n=1}^{\infty}\left|\langle(T_k - r_k)x_{n_1}, x_n\rangle\right|^2 \leq \left\|(T_k - r_k)x_{n_1}\right\|^2$$

and

$$\sum_{n=1}^{\infty}\left|\langle(T_k - r_k)x_n, x_{n_1}\rangle\right|^2 \leq \left\|(T_k - r_k)^* x_{n_1}\right\|^2$$

Thus, by Bessels inequality, there is an integer $n_2 > n_1$ such that

$$\sum_{n=n_2}^{\infty}\left|\langle(T_k - r_k)x_{n_1}, x_n\rangle\right|^2 < 2^{-1}$$

and

$\sum_{n=n_2}^{\infty}\left|\langle(T_k - r_k)x_n, x_{n_1}\rangle\right|^2 < 2^{-1}$.

If this procedure is repeated, a strictly increasing sequence $\{n_t\}_{t=1}^{\infty}$ of positive integers is obtained such that we have

$$\sum_{n=n_{t+1}}^{\infty}|\langle(T_k - r_k)x_t, x_n\rangle|^2 < 2^{-t}$$

And

$$\sum_{n=n_{t+1}}^{\infty}\left|\langle(T_k - r_k)x_n, x_{n_t}\rangle\right|^2 < 2^{-t} \quad (**)$$

$(*)$ and $(**)$ both imply that

$$\sum_{t,l=1}^{\infty}\left|\langle(T_k - r_k)x_t, x_{n_l}\rangle\right|^2 < \infty \quad (***)$$

If P is an orthogonal projection onto the subspace $\mathcal{M}$ spanned by $x_{n_1}, x_{n_2}, \ldots$, then

$\sum_{t,l=1}^{\infty}\left|\langle(PT_kP - r_kP)x_{n_t}, x_{n_l}\rangle\right|^2 = \sum_{t,l=1}^{\infty}\left|\langle(T_k - r_k)x_{n_t}, x_{n_l}\rangle\right|^2 < \infty$ by $(***)$, hence PT_kP is a Hilbert - Schmidt operator and therefore $PT_kP - r_kP \in K(X)$.

We then show that (3) implies (2).

Let $\{x_n\}_{n=1}^{\infty} \in X$ be a sequence of vectors converging weakly to $0 \in X$ such that $\langle T_kx_n, x_n\rangle \to r_k$ and $\|T_kx_n\| \to \|T_k\|_e$; $1 \leq k \leq m$.

Construct an orthonormal sequence $\{y_n\}_{n=1}^{\infty}$ such that $\|T_ky_n\| \to \|T_k\|_e - \frac{1}{n}$ and $|(T_ky_n, y_n)| < \frac{1}{n}$ as follows.

Suppose that the set $\{x_1, \ldots, x_n\}$ has been constructed. Let $\mathcal{M}$ be the subspace spanned by $x_1, \ldots, x_n$ and P be the projection onto $\mathcal{M}$. Then we have

$\|Px_n\| \to 0$ as $n \to \infty$. Let

$z_n = \|(I - P)x_n\|^{-1}\big((I - P)x_n\big)$.

We have

$T_kz_n = \|(I - P)x_n\|^{-1}(T_k(I - P)x_n)$.

This gives

$\langle T_k z_n, z_n\rangle = \langle \|(I-P)x_n\|^{-1}(T_k(I-P)x_n), \|(I-P)x_n\|^{-1}(T_k(I-P)x_n)\rangle = \|(I-P)x_n\|^{-2}\{\langle T_k x_n, x_n\rangle - \langle T_k x_n, Px_n\rangle - \langle T_k Px_n, Px_n\rangle\} \to r_k$

We choose n large enough such that $|\langle T_k z_n, z_n\rangle - r_k| < (n+1)^{-1}$. If we let $z_n = x_{n+1}$ we get $|\langle T_k x_{n+1}, x_{n+1}\rangle r_k| < (n+1)^{-1}$. To show that (3) implies (1). Suppose that for a point $r_k \in \mathbb{C}^m$ there is a sequence $\{x_n\} \in X$ such that $\langle T_k x_n, x_n\rangle \to r_k$

Since every sequence $\{x_n\} \to 0$ weakly, and $\|x\|= 1$, we have $r_k \to MaxW_{em}(T)$.

Recall that a subset C of a linear space M is convex if $\forall x, y \in C$ the segment joining x and y is contained in C, that is, $tx + (1-t)y \in C \ \forall\, t \in [0,1]$.

A set S is star-shaped if $\exists\, y \in S$ such that $\forall\, x \in S$ the segment joining x and y is contained in S, that is $\lambda x + (1-\lambda)y \in S \ \forall\, \lambda \in [0,1]$.

A point $y \in S$ is a star center of S if there is a point $x \in S$ such that the segment joining x and y is contained in S.

Starshapedness is related to convexity in that a convex set is starshaped with all its points being star centers.

Theorem 1.4. *Suppose $T = (T_1, \dots, T_m) \in B(X)$. Then $MaxW_{em}(T)$ is nonempty, compact and each element $r \in MaxW_{em}(T)$ is a star center of $MaxW_m(T)$. Moreover, $MaxW_{em}(T)$ is convex.*

Proof. First, we prove that $MaxW_{em}(T)$ is nonempty. To do this, from first theorem, there exists an orthonormal sequence $\{x_n\}_{n=1}^{\infty} \in X$ such that

$\langle T_k x_n, x_n\rangle \to r_k$ and $\|T_k x_n\| \to \|T_k\|_e$;$1 \le k \le m$.

Thus the sequence $\{\langle T_k x_n, x_n\rangle\}_{n=1}^{\infty}$ is bounded. Choose a subsequence and assume that $\langle T_k x_n, x_n\rangle$ converges. Then $MaxW_{em}(T)$ is nonempty.

The compactness of $MaxW_{em}(T)$ can be seen right from its properties.

$MaxW_{em}(T) = MaxW_{em}(T+K) \subseteq MaxW_m(T+K) : K \in \mathcal{K}(X)$ where $\mathcal{K}(X)$ denote the sets of compact operators in $B(X)$. Since $MaxW_m(T+K)$ is compact, the joint essential numerical range is also compact.

To prove that each element $r \in MaxW_{em}(T)$ is a star center of $MaxW_m(T)$ it should be shown that $(1-\lambda)P + \lambda r \in MaxW_m(T) : \lambda \in [0,1]$ where $r \in MaxW_{em}(T)$ and $p \in MaxW_m(T)$. Assume without loss of generality that $\|T\| = 1$. Suppose $s \in MaxW_m(T)$ so that $s = \lambda r + (1-\lambda)p$. Let $\{x_n\}$ and be orthonormal sequences in X such that $r = \langle Tx_n, x_n\rangle, p = \langle Te_n, e_n\rangle$

and $\|x_n\| = \|e_n\|=1$

Then,

$$s = \lambda\langle Tx_n, x_n\rangle + (1-\lambda)\langle Te_n, e_n\rangle$$

$$= \langle T\sqrt{\lambda}x_n, \sqrt{\lambda}x_n\rangle + \langle T\sqrt{1-\lambda}e_n, \sqrt{1-\lambda}e_n\rangle$$

$$= \langle (T\sqrt{\lambda}x_n + T\sqrt{1-\lambda}e_n), (\sqrt{\lambda}x_n + \sqrt{1-\lambda}e_n)\rangle$$

$$\|\sqrt{\lambda}x_n + \sqrt{1-\lambda}e_n\| =^2 \left(\|\sqrt{\lambda}x_n\|^2 + \sqrt{1-\lambda}e_n\right)$$

$$= \lambda\|x_n\|^2 + (1-\lambda)\|e_n\|^2$$

$$= \lambda + (1-\lambda) = 1$$

Thus, $(1-\lambda)r + \lambda p \in MaxW_m(T)$.

Convexity of $MaxW_{em}(T)$ is proved by showing that for $r, p \in MaxW_{em}(T)$ and $\lambda \in [0; 1]$ we have $\lambda r + (1-\lambda)p \in MaxW_{em}(T)$ Now, $r \in MaxW_{em}(T) = MaxW_{em}(T+k)$ for every $K \in \mathcal{K}(X)$ and $p \in MaxW_{em}(T) = MaxW_{em}(T+K) \subseteq MaxW_{m}(T+K)$. From Theorem above, $\lambda r + (1-\lambda)p \in MaxW_{m}(T+K)$. Thus, $\lambda r + (1-\lambda)p \in \cap \{MaxW_{m}(T+K) \colon K \in \mathcal{K}(X)\} = MaxW_{em}(T)$. Hence $MaxW_{em}(T)$ is convex.

3. CONCLUSION

We state the following theorem without proof.

Proposition 1.5. *Let* $T \in B(X)$. *If* $\|T_k x\| \neq \|T_k\| \|x\|$, *then* $MaxW_{em}(T) = MaxW_{m}(T)$.

References

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