On the Existence of Almost Affinely Flat Structure

Induced by Hypersurface Immersion on

Connected Compact Manifold

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Abstract

Given a hypersurface immersion and a transversal vector field, the formula of Gauss leads to an induced connection and a symmetric bilinear function called affine fundamental form. We define the norm of tensor field using the affine fundamental form (assumed to be nondegenerate) and prove that a hypersurface immersion on a connected compact n-dimensional differential manifold M into the affine space $\mathbb{R}^{n+1}$ induces an almost affinely flat structure on M.

Keywords: Affine (hypersurface) immersion, curvature tensor, almost affinely flat, Riemannian metric

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1. Introduction

In [7] and [12], almost flat manifolds (in the sense of Gromov) are classified. This is a subject which has been on much scrutiny in differential geometry ([3], [4], [10] and references therein). This concept has been generalized to affine differential geometry and it is proved in [1] that the three-dimensional sphere is almost affinely flat. In general, a manifold which is almost flat is necessarily almost affinely flat, but the converse is not true. So far, there is no classification of almost affinely flat manifolds. Nevertheless, the Pontryagin number serve as the obstruction to the existence of almost affinely flat connections on manifolds of the form M^{4n}.

On the other hand, the Geometry of Affine Immersion studies the nature of geometrical "objects" induced by an affine immersion. Suppose a differential manifold M, not provided with any particular affine connection, is immersed into a differential manifold $\widetilde{M}$ equiped with an affine connection $\widetilde{\nabla}$ and choose any transversal vector field. Then intuitively the immersion transfers affinely some characteristics from $\widetilde{M}$ into M.

With the theory of affine differential geometry developed in [11], it is well known that if we consider an immersion and we take any transversal vector field, there is the decomposition of Gauss which gives a torsion-free induced connection and a symmetric bilinear function, namely h, called the affine fundamental form. Recently [6], [8] and [9] studied the case where $\widetilde{M}$ is a complex space form and obtained various results such as the characterizations of (connected) Hopf hypersurfaces. Immersions into affine space was studied in [13] and as results, there is no ovaloid (See Definition 7.2 of Chapter III in [11]) with vanishing unimodular mean curvature. In 2017, [5] studied the differential geometry of immersed surfaces in three-dimensional normed space and proved that under additional hypothesis, a connected compact surface without boundary whose Minkowski Gaussian curvature is constant must be Minkowski sphere. Now, in the case where $\widetilde{M}$ is the affine space $\mathbb{R}^{n+1}$ and $\widetilde{\nabla}$ is the usual flat affine connection D, we aim to investigate whether there is a transversal vector field such that the induced connection ∇ on the connected compact manifold M is almost affinely flat.

2. Preliminaries

In this section, we denote M – the n-dimensional differential manifold ($n \geq 2$), $T_x(M)$ – Tangent space at $x \in M$, ∇ – affine connection on M and D – usual flat affine connection on the affine space $\mathbb{R}^{n+1}$. We assume that all vector fields as well as the manifolds are smooth.

Definition 2.1. The torsion of ∇ is a tensor field of type $(1,2)$

$$\mathrm{T} : T(M) \times T(M) \to T(M)$$
$$(X, Y) \mapsto T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

where $[\,,]$ is the Lie algebra bracket.
The affine connection is called *torsion-free* or *symmetric* if $\mathrm{T} = 0$.

Definition 2.2. The curvature of ∇ is a tensor field of type (1,3)

$$R \; : \; T(M) \times T(M) \times T(M) \longrightarrow T(M)$$
$$(X, Y, Z) \longmapsto R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

Note: In local coordinates on $T_x(M)$, the components R^i_{jkl} of R are given by

$$R(\partial_k, \partial_l)\partial_j = \sum_i R^i_{jkl} \partial_i$$

A connection is said to be *flat* if both the torsion tensor and the curvature tensor identically vanish.

Definition 2.3. A *nondegenerate metric* (also called *pseudo-Riemannian metric*) is a tensor field of type (0,2), $g \; : \; T(M) \times T(M) \longrightarrow \mathbb{R}$ satisfying the following conditions :

i) $g(X, Y) = g(Y, X)$; (Symmetry)
ii) $g(X, Y) = 0$, $\forall Y \in T_x(M) \Longrightarrow X = 0$ (Nondegeneracy).

Definition 2.4. A *Riemannian metric* is a tensor field of type (0,2), $g \; : \; T(M) \times T(M) \longrightarrow \mathbb{R}$ satisfying the following conditions :

a) $g(X, Y) = g(Y, X)$; (Symmetry)
b) $\forall X \neq 0, g(X, X) > 0$ (Positive-definite).

Remark 2.1. It is easy to prove that the condition $b)$ in Definition 2.4 implies the condition $ii)$ in Definition 2.3. In other words, a Riemannian metric is a symmetric (nondegenerate) positive-definite tensor field of type (0,2) on M.

Definition 2.5. Let $x = (x^1, \dots, x^n) \in M$ and denote $\partial_i = \frac{\partial}{\partial x^i}$ $(i = 1, \dots, n)$ as local coordinate system on $T_x(M)$. If a nondegenerate metric h satisfies $h(\partial_i, \partial_j) = \epsilon_i \delta_{ij}$; $\epsilon_i = \pm 1$ where δ_{ij} is the Kronecker delta, we say that $\{\partial_1, \dots, \partial_n\}$ is an orthonormal basis of $T_x(M)$ relative to h. We also say that $\{\partial_1, \dots, \partial_n\}$ is an h-orthonormal basis of $T_x(M)$.

Lemma 2.1.

Let h be a nondegenerate metric and ϕ any positive function. Consider $\overline{h} = \frac{1}{\phi} h$. Then

(a) $\overline{h}$ is a nondegenerate metric ;
(b) $\mathcal{B} = \{\partial_1, \dots, \partial_n\}$ is an h-orthonormal basis of $T_x(M)$ if and only if $\overline{\mathcal{B}} = \{\sqrt{\phi}\partial_1, \dots, \sqrt{\phi}\partial_n\}$ is an $\overline{h}$-orthonormal basis of $T_x(M)$.
(c) h and $\overline{h}$ have the same signature with respect to $\mathcal{B}$ and $\overline{\mathcal{B}}$ respectively. In particular, if h is positive-definite with respect to $\mathcal{B}$ then $\overline{h}$ is positive-definite with respect to $\overline{\mathcal{B}}$, and the converse is also true.

Proof :
    (a) The proof uses the condition that h is nondegenerate.

(b) & (c) For any $i, j = 1, \dots, n$

$$h(\partial_i, \partial_j) = \epsilon_i \delta_{ij} \iff \phi \overline{h}(\partial_i, \partial_j) = \epsilon_i \delta_{ij}$$
$$\iff \left(\sqrt{\phi}\right)^2 \overline{h}(\partial_i, \partial_j) = \epsilon_i \delta_{ij}$$
$$\iff \overline{h}(\sqrt{\phi}\partial_i, \sqrt{\phi}\partial_j) = \epsilon_i \delta_{ij} \qquad \blacksquare$$

Proposition 2.1. For a hypersurface immersion $f : M \longrightarrow (\mathbb{R}^{n+1}, D)$, suppose we have a transversal vector field ξ on M. Then there is a torsion-free induced connection ∇ satisfying the *formula of Gauss* :
$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X, Y)\xi$$
where h is a symmetric bilinear function on the tangent space $T_x(M)$.
If that equation holds, then $f : (M, \nabla) \longrightarrow (\mathbb{R}^{n+1}, D)$ is called an *affine immersion*.

Note: We shall assume in this paper that h is nondegenerate and can be treated as a pseudo-Riemannian metric. We also say that f is a *nondegenerate hypersurface immersion* and M is a *nondegenerate hypersurface*.
    In addition we have the following decomposition
$$D_X \xi = -f_*(SX) + \tau(X)\xi$$
called the *formula of Weingarten*.
The $(1,1)$-tensor S is called the (affine) *shape operator* and the 1-form τ is called the *transversal connection form*.

Remark 2.1. In the case of a connected compact nondegenerate hypersurface M, there exists a transversal vector field such that the affine metric h is positive-definite at every point of M. (See Proposition 7.3 of Chapter III in [11]).
The next theorem gives the relationship between the curvature R of the induced connection, the affine fundamental form h and the shape operator S.

Theorem 2.1.
For an arbitrary transversal vector field ξ the induced connection ∇, the affine fundamental form h and the shape operator S satisfy the following equation:
$$R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY$$
Proof : (See [11], page 33)

From the equation in Theorem 2.1 above, under the hypothesis that the affine fundamental form h is nondegenerate, we also state the following:

Theorem 2.2.
The shape operator S and the curvature R of the induced connection satisfy:
$$S = 0 \quad \text{if and only if} \quad R = 0$$

Proof :
Obviously $S = 0$ implies $R = 0$ by the Theorem 2.1. Assume that $R = 0$ and Let $\{\partial_1, \dots, \partial_n\}$ be an h-orthonormal basis. For any $i = 1, \dots, n$, take $k \neq i$. From the

assumption $R = 0$, we get
$$R(\partial_i, \partial_k)\partial_k = 0$$

Again, using the Theorem 2.1, we obtain
$$h(\partial_k, \partial_k)S\partial_i - h(\partial_i, \partial_k)S\partial_k = 0$$
$$\Longrightarrow \quad \epsilon_k S\partial_i = 0$$
$$\Longrightarrow \quad \sum_j S_i^j \partial_j = 0$$
$$\Longrightarrow \quad S_i^j = 0$$

So that all the components of S are equal to 0. ∎

Next we consider a change of a transversal vector field for a given immersion f.

Theorem 2.3.

Suppose we change a transversal vector field ξ to
$$\overline{\xi} = \phi\xi + f_*(Z)$$
where Z is a tangent vector field on M, and ϕ is nonvanishing function. Then the affine fundamental form changes to :
$$\overline{h} = \frac{1}{\phi}h$$
and the shape operator changes as follow :
$$\overline{S} = \phi S - \nabla Z + \overline{\tau}(.)Z$$
In the particular case where $Z = 0$, this can be written as $\overline{S} = \phi S$.

Proof : (See [11], page 36)

Corollary 2.1. Suppose that there is a transversal vector field ξ such that the affine fundamental form h is definite. Then there exists a transversal vector field $\tilde{\xi}$ such that the affine fundamental form $\tilde{h}$ is positive-definite.

Proof :

If h is positive-definite, then $\tilde{\xi} = \xi$.

If h is negative-definite, then we take $\tilde{\xi} = -\xi$ and we obtain that the affine fundamental relative to $\tilde{\xi}$ is given by $\tilde{h} = -h$ which is positive-definite.

3. Main results

We recall the definition of almost affinely flat manifold stated in [1].

Definition 3.1. A Riemannian manifold (M, g) is called *almost affinely flat* if for any positive real number ε, there exists a torsion-free connection ∇ such that the norm of the curvature of ∇ satisfies $\|R\| < \varepsilon$ at every point of M, where the norm $\|R\|$ is determined by the metric g.

(Explicitly, $\|R\|^2 = \frac{1}{2}\sum_{ijkl}\left(R_{kij}^l\right)^2$, where $R(X_i, X_j)X_k = \sum_l R_{kij}^l X_l$ and $\{X_1, \dots, X_n\}$ is an g-orthonormal basis of $T_x M$.)

Remark 3.1. On a compact manifold, the notion does not depend on the metric, because the norms of tensor fields with respect to different metrics are within constants of each other [1].

In our case, a priori the manifold M is not provided with any metric. Meanwhile M is connected compact nondegenerate hypersurface and we shall use the affine fundamental form h considered as a positive-definite metric.

Now, our main result on connected compact manifold is stated as follows.

Theorem 3.1.

Given a nondegenerate hypersurface immersion $f : M \longrightarrow (\mathbb{R}^{n+1}, D)$ on a connected compact manifold, then for any positive real number ε there is a transversal vector field $\xi = \xi(\varepsilon)$ such that (M, ∇) is almost affinely flat, where ∇ is the affine connection induced by f and ξ.

Proof :

Let ξ be a transversal vector field such that the affine fundamental form h is positive-definite. We already know that the induced connection is torsion-free (Proposition 2.1). Next, recall from Theorem 2.1 that

$$R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY$$

where R is the curvature tensor of ∇.

In local coordinate system, $R(\partial_i, \partial_j)\partial_k = h(\partial_j, \partial_k)S\partial_i - h(\partial_i, \partial_k)S\partial_j$.

If $\mathcal{B} = \{\partial_1, \dots, \partial_n\}$ is an orthonormal basis of $T_x(M)$ relative to h then the components of R are either 0 or S_i^j, where S_i^j are the components of the shape operator S (that is $S\partial_i = \sum_j S_i^j \partial_j$).

For example:

$R(\partial_i, \partial_i)\partial_k = h(\partial_i, \partial_k)S\partial_i - h(\partial_i, \partial_k)S\partial_i = 0$; which leads to $R_{jkk}^i = 0$, for all i, j, k.

Also $R(\partial_i, \partial_j)\partial_i = h(\partial_j, \partial_i)S\partial_i - h(\partial_i, \partial_i)S\partial_j = -S\partial_j$; for all $i \neq j$, from which we get $R_{iij}^k = -S_j^k$ for all $i \neq j$.

Therefore the square norm of R is the half of a combination of the square of the components of the shape operator S with coefficients equal to 1. Also the number of terms in that combination is less than n^4. Indeed, the curvature tensor has n^4 components and from the computations above, some of these components are equal to 0.

Now, we are in a position to prove the Theorem 3.1.

If the shape operator $S = 0$ then there is nothing to prove (Theorem 2.2).

Otherwise, let $c = \max_{i,j}\left\{\left(S_i^j\right)^2\right\}$ and consider the change $\overline{\xi} = \phi\xi$ with $\phi = \frac{\varepsilon}{n^2}\sqrt{\frac{2}{c}}$.

(*Notation*: The objects h, S and R for $\overline{\xi}$ will be denoted by $\overline{h}$, $\overline{S}$ and $\overline{R}$.)

This change of transversal vector field gives $\overline{h} = \frac{1}{\phi}h$ (Theorem 2.3).

From Lemma 2.1, we obtain that $\overline{\mathcal{B}} = \{\sqrt{\phi}\partial_1, \ldots, \sqrt{\phi}\partial_n\}$ is an orthonormal basis relative to $\overline{h}$. Furthermore $\overline{h}$ is also positive-definite (Riemannian metric).

However such change of the basis does not affect the components of the shape operator S. Indeed, the components $\tilde{S}_i^j$ of S relative to the basis $\overline{\mathcal{B}}$ are given by

$$S\overline{\partial}_i = \sum_j \tilde{S}_i^j \overline{\partial}_j$$

where $\overline{\partial}_i = \sqrt{\phi}\partial_i$.

But using the linearity of S, we get the following :

$$\begin{aligned}
S(\sqrt{\phi}\partial_i) &= \sqrt{\phi}S\partial_i \\
&= \sqrt{\phi}\sum_j S_i^j \partial_j \\
&= \sum_j S_i^j \sqrt{\phi}\partial_j
\end{aligned}$$

Thus $\tilde{S}_i^j = S_i^j$.

Again from Theorem 2.3, we know that $\overline{S} = \phi S$ (because $Z = 0$). Thus the components of $\overline{S}$ relative to the basis $\overline{\mathcal{B}}$ are given by

$$\overline{S}_i^j = \phi \tilde{S}_i^j = \phi S_i^j$$

which implies that

$$\left(\overline{S}_i^j\right)^2 = \phi^2 \left(S_i^j\right)^2 = \frac{2\varepsilon^2}{n^4 c}\left(S_i^j\right)^2 \leq \frac{2\varepsilon^2}{n^4}$$

Therefore $\left\|\overline{R}\right\|^2 < \varepsilon^2$. ∎

The Theorem 3.2 below is a generalization of the Theorem 3.1. We consider the general case in which M is not necessarily nondegenerate compact connected.

Theorem 3.2.
Let $f : M \longrightarrow (\mathbb{R}^{n+1}, D)$ be a hypersurface immersion. Suppose that there exists a transversal vector ξ such that the affine fundamental form h, is definite. Then there exists a transversal vector field ξ' such that (M, ∇') is almost affinely flat, where ∇' is the affine connection induced by f and ξ'.
Proof :
From Corollary 2.1, there exists a transversal vector field such that the affine fundamental form is positive-definite. The remaining part of the proof follows easily.

4. Conclusion and Suggestions

In this paper, we obtained that under the hypothesis of nondegenerescence, a compact connected hypersurface is almost affinely flat. This result has been generalized to any arbitrary smooth manifold for which there exists a definite affine fundamental form.

The general definition of almost affinely k-flat needs the curvature tensor R to satisfy $\left\|\nabla^i R\right\| < \varepsilon$, for $i = 0, 1, \dots, k$ with the convention $\nabla^0 R \equiv R$ (See [2]). This paper deals with almost affinely 0-flat. However, we do not know whether there is a transversal vector field such that the induced connection is almost affinely k-flat for $k \geq 1$. Also, it is interesting to determine the existence or non-existence of equiaffine transversal field which leads to an almost affinely flat connection. Again, it is proved in Proposition 7.3 of Chapter III in [11] that there is a global Blaschke normal field provided on compact connected hypersurface M such that the affine metric is positive-definite. But it is an open problem to investigate the existence of a Blaschke normal field inducing an almost affinely flat structure on M.

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