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A Special Nörlund Means and its Spectrum

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Abstract

In summability theory, different classes of matrices have been investigated and characterized. There are various types of summability methods e.g. *Nörlund* means, Cesaro, Riesz, Euler, Abel and many others. The spectrum of an operator plays a crucial role in the development of Tauberian theory for the operator and Mercerian theorems which are used to determine the limit or sum of a convergent sequence or series.

In this paper, the spectrum and eigenvalues of a special *Nörlund* matrix as a bounded operator on the sequence space c is investigated and determined. This is achieved by applying Banach space theorems of functional analysis as well as summability methods of summability theory. It is shown that the spectrum consist of the set $\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{3}\right| \leq \frac{1}{3}\} \cup \{1\}$.

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Keywords: *Nörlund* Means, Spectrum, Eigenvalues, convergence, Operator

Notations

$\mathbb{R}^+$ - will denote the set of positive real numbers;$\mathbb{R}$ - the set of real numbers;$\mathbb{C}$ - the set of complex numbers;$\|.\|$- norm of; $\rightarrow$- tend to; $\emptyset$- empty set; c- the set of all sequences which converge; $\ell_p(0 < p < \infty)$ - sequences such that $\sum_{k=0}^{\infty} |x|^p < \infty$.

1 Introduction

Nörlund mean matrix is an infinite triangular matrix $A = (a_{nk})$ with

$$a_{nk} = \begin{cases} \frac{p_{n-k}}{P_n} & 0 \leq k \leq n \\ 0, & k > n \end{cases} \tag{1}$$

, where $p_0 > 0$, $p_k \geq 0$ for all $k \geq 1$ and $P_n = \sum_{k=0}^{n} p_k$. In this paper we let $p_0 = p_1 = p_2 = m$, $p_3 = p_4 = p_5 \cdots = 0, m \in \mathbb{Z}$, then the matrix is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \cdots \\ & & \cdots & & \end{pmatrix} \tag{2}$$

Theorem 1.1. *$A \in (c, c)$ if and only if*

i. $\lim_{n\to\infty} a_{nk} = a_k$ for each fixed k, $k = 0, 1, 2, \ldots$

ii. $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = a$ as $n \to \infty$

iii. $\sup_{n\geq 0}\{\sum_{k=0}^{\infty} |a_{nk}|\} < \infty$, [7].

Theorem 1.2. *$A \in (l_1, l_1)$ if and only if*

i. $\sum_{n=0}^{\infty} |a_{nk}| < \infty$ for each fixed k

ii. $\sup_k\{\sum_{n=0}^{\infty} |a_{nk}|\} < \infty$, [2].

Definition 1.3. (Adjoint Operator T^*)

The adjoint $T^*$of linear operator $T \in B(X, Y)$ is the mapping from $Y^*$to X^*defined by $T^* \circ f = f \circ T$, $f \in Y^*$

Theorem 1.4. *T^* is linear and bounded. Moreover, $\|T^*\| = \|T\|$, [4].*

Theorem 1.5. *A linear Operator $T \in B(X, Y)$ has a bounded inverse T^{-1} defined on all Y if and only if its adjoint T^*has a bounded inverse $(T^*)^{-1}$ defined on all of X^*. When these inverses exist, $(T^{-1})^* = (T^*)^{-1}$, [8, 10].*

Definition 1.6. (Resolvent Operator, $R_\lambda = (T - \lambda I)^{-1}$)

Let X be a non - empty Banach space and suppose that $T : X \to X$. With T, associated is the operator $T_\lambda = T - \lambda I$, $\lambda \in \mathbb{C}$, where I is the identity operator on X. If $T_\lambda = T - \lambda I$ has an inverse, then it is denoted by $R_\lambda(T)$ or simply R_λ and call it the resolvent operator of T.

Definition 1.7. (Resolvent set $\rho(T)$, spectrum $\sigma(T)$)

Let X be a non - empty Banach space and suppose that $T : X \to X$. The resolvent set $\rho(T)$ of T is the set of complex numbers $\lambda$for which $(T - \lambda I)^{-1}$ exist as a bounded operator with the domain X. The spectrum $\sigma(T)$ of T is the compliment of $\rho(T)$ in $\mathbb{C}$.

Theorem 1.8. *The resolvent set $\rho(T)$ of a bounded linear operator T on a Banach space X is open; hence the spectrum $\sigma(T)$ of T is closed, [1, 4].*

Theorem 1.9. *If X is any Banach space and $T \in B(X)$, then $\sigma(T) \neq \emptyset$, [1, 4].*

The spectrum $\sigma(T)$ of a bounded linear operator $T : X \to X$ on a Banach space X is compact and lies in the disk given by:

$$|\lambda| = \|T\| \tag{3}$$

[4].

Theorem 1.10. *Let $T \in B(X)$, where X is any Banach space, then the spectrum of $T^*$is identical to the spectrum of T. Furthermore, $R_\lambda(T^*) = (R_\lambda(T))^*$for $\lambda \in \rho(T) = \rho(T^*)$, [8] and [10].*

2 The spectrum of A operator on c

We determine the spectrum of matrix A as an operator on c The matrix $A =$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \cdots \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ & & \cdots & & & \end{pmatrix}$$

Corollary 2.1. $A \in B(c)$

Proof. i. $\lim\limits_{n\to\infty} a_{nk} = 0$ for each fixed k, $k = 0, 1, 2, \ldots$ □

ii. $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{\infty} a_{nk} = 1$ as $n \to \infty$ iii. $\|A\| = \sup_{n\geq 0}\{\sum\limits_{k=0}^{\infty} |a_{nk}|\} = 1 < \infty$

2.1 The Eigenvalues of A operator on c

Theorem 2.2. *The eigenvalue of $A \in B(c)$ is the singleton set* $\{1\}$

Proof. Solving the system $Ax = \lambda x$, $x \neq \theta$ in c and $\lambda \in \mathbb{C}$, then

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \cdots \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ & & \cdots & & & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{pmatrix} \tag{4}$$

which gives

$$\begin{gathered} x_0 = \lambda x_0 \\ \tfrac{1}{2}x_0 + \tfrac{1}{2}x_1 = \lambda x_1 \\ \tfrac{1}{3}x_0 + \tfrac{1}{3}x_1 + \tfrac{1}{3}x_2 = \lambda x_2 \\ \tfrac{1}{3}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{3}x_3 = \lambda x_3 \\ \vdots \\ \tfrac{1}{3}x_{n-2} + \tfrac{1}{3}x_{n-1} + \tfrac{1}{3}x_n = \lambda x_n \\ \vdots \end{gathered} \tag{5}$$

Solving equation 5, if x_0 is the first non zero entry of x, then $\lambda = 1$. But $\lambda = 1$ implies $x_0 = x_1 = x_2 = \cdots = x_n = \cdots$, which shows that x is in the span of $\delta = (1, 1, 1, 1, \cdots)$ which tends to 1 as n tends to infinity. Therefore $\lambda = 1$ is an eigenvalue of $A \in B(c)$. When x_1 is the first non zero entry of x, $\lambda = \frac{1}{2}$. But $\lambda = \frac{1}{2}$ implies $x_0 = 0$, $x_2 = 2x_1$, $x_3 = 6x_1$, $x_4 = 16x_1$, $x_5 = 44x_1$, $\cdots$

which shows that x is spanned by $\{0, 1, 2, 6, 16, 44, \cdots\}$ an increasing sequence which is not bounded above, hence does not converge as n tends to infinity. If x_{n+2} is the first non zero entry for $n = 0, 1, 2, 3, \cdots$, then $\lambda = \frac{1}{3}$, solving the system gives $x_n = 0$ for $n = 0, 1, 2, 3, \cdots$ which is a contradiction hence $\lambda = \frac{1}{3}$ cannot be an eigenvalue. □

2.2 The Eigenvalues of A^* operator on ℓ_1

Theorem 2.3. *Let $A : c \to c$ be a linear map and define $A^* : c^* \to c^*$ i,e $A^* : \ell_1 \to \ell_1$ by $A^*(g) = g \circ A,\ g \in c^* \equiv \ell_1$. Then both A and A^* must be given by a matrix. Moreover $A^* : \ell_1 \to \ell_1$ is given by the matrix,*

$$A^* = \begin{pmatrix} \chi(\lim A) & (v_n)_0^\infty \\ (a_k)_0^\infty & A^t \end{pmatrix} \tag{6}$$

[5].

Corollary 2.4. *Let $A : c \to c$. Then $A^* \in B(\ell_1)$ and*

$$A^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & \frac{1}{2} & \frac{1}{3} & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \cdots \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \cdots \\ & & & \cdots & & \end{pmatrix} \tag{7}$$

Proof. By theorem 2.3

$$A^* = \begin{pmatrix} \chi(\lim A) & (v_n)_0^\infty \\ (a_k)_0^\infty & A^t \end{pmatrix} \tag{8}$$

where $\chi(\lim A) = \lim_A(\delta) - \sum\limits_{k=0}^{\infty} \lim_A \delta^k$ is called the characteristic of a matrix A $\delta = \{1, 1, 1, 1, \ldots\}$, $\lim_A(\delta) = 1$ and $\delta^k = \{0, 0, 0, 0, \ldots, 1, 0, 0 \ldots\}$, having zeros with 1 in the k^{th}position, $\lim_A \delta^k = 0$ and $\sum \lim_A \delta^k = 0$. Hence $\chi(\lim A) = 1 - 0 = 1$

$v_n = \chi(P_n \circ T) = (P_n \circ T)\delta - \sum\limits_{k=0}^{\infty} (P_n \circ T)\delta^k$ but for matrix A, $(P_n \circ T)\delta = 1,\ \forall n$ and $\sum\limits_{k=0}^{\infty} (P_n \circ T)\delta^k = 1$ i.e

$$\begin{gathered} v_0 = 1 - (1 + 0 + 0 + 0 + \cdots) = 1 - 1 = 0 \\ v_1 = 1 - \left(\tfrac{1}{2} + \tfrac{1}{2} + 0 + 0 + \cdots\right) = 1 - 1 = 0 \\ v_2 = 1 - \left(\tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{3} + 0 + \cdots\right) = 1 - 1 = 0 \\ v_3 = 1 - \left(0 + \tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{3} + 0 + 0 \cdots\right) = 1 - 1 = 0 \\ \vdots \\ v_n = 0,\ \ n \geq 0 \end{gathered} \tag{9}$$

hence the matrix becomes

$$\begin{pmatrix} 1 & \theta \\ \theta & A^T \end{pmatrix} \tag{10}$$

□

Theorem 2.5. *The eigenvalues of* $A^* \in B(\ell_1)$ *is the set* $\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{3}\right| < \frac{1}{3}\}$

Proof. Consider the system $A^*x = \lambda x$, $x \neq \theta$ in ℓ_1 and $\lambda \in \mathbb{C}$,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & \frac{1}{2} & \frac{1}{3} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \cdots \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \cdots \\ & & & \cdots & & & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ \vdots \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ \vdots \end{pmatrix} \tag{11}$$

which gives

$$\begin{gathered} x_0 = \lambda x_0 \\ x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{3}x_3 = \lambda x_1 \\ \tfrac{1}{2}x_2 + \tfrac{1}{3}x_3 + \tfrac{1}{3}x_4 = \lambda x_2 \\ \tfrac{1}{3}x_3 + \tfrac{1}{3}x_4 + \tfrac{1}{3}x_5 = \lambda x_3 \\ \tfrac{1}{3}x_4 + \tfrac{1}{3}x_5 + \tfrac{1}{3}x_6 = \lambda x_4 \\ \cdots \\ \tfrac{1}{3}x_{n-2} + \tfrac{1}{3}x_{n-1} + \tfrac{1}{3}x_n = \lambda x_{n-2}, \; for\, n \geq 5 \end{gathered} \tag{12}$$

, solving the system gives

$x_n = 3(\lambda - \frac{1}{3})x_{n-2} - x_{n-1}$, $n \geq 5$, or

$$for\, n\, odd, \quad x_n = 3^{\frac{n-1}{2}-1}(\lambda - \tfrac{1}{3})^{\frac{n-1}{2}-1}x_3 - \sum_{k=0}^{\frac{n-1}{2}-2} 3^k(\lambda - \tfrac{1}{3})^k x_{n-(2k+1)}, \; n \geq 5$$

$$for\, n\, even, \quad x_n = 3^{\frac{n}{2}-2}(\lambda - \tfrac{1}{3})^{\frac{n}{2}-2}x_4 - \sum_{k=0}^{\frac{n}{2}-3} 3^k(\lambda - \tfrac{1}{3})^k x_{n-(2k+1)} \;\; n \geq 6$$

Hence $\sum\limits_{n=0}^{\infty} |x_n| = |x_0|+|x_1|+|x_2|+|x_3|+|x_4| + \sum\limits_{\substack{n=6 \\ n\, even}}^{\infty} \left| 3^{\frac{n}{2}-2}(\lambda - \frac{1}{3})^{\frac{n}{2}-2}x_4 - \sum\limits_{k=0}^{\frac{n}{2}-3} 3^k(\lambda - \frac{1}{3})^k x_{n-(2k+1)} \right| +$

$\sum\limits_{\substack{n=5 \\ n\, odd}}^{\infty} \left| 3^{\frac{n-1}{2}-1}(\lambda - \frac{1}{3})^{\frac{n-1}{2}-1}x_3 - \sum\limits_{k=0}^{\frac{n-1}{2}-2} 3^k(\lambda - \frac{1}{3})^k x_{n-(2k+1)} \right|$

$\leq \sum\limits_{n=0}^{4} |x_n| + \sum\limits_{\substack{n=6 \\ n\, even}}^{\infty} \left| 3^{\frac{n}{2}-2}(\lambda - \frac{1}{3})^{\frac{n}{2}-2}x_4 \right| + \sum\limits_{\substack{n=5 \\ n\, odd}}^{\infty} \left| 3^{\frac{n-1}{2}-1}(\lambda - \frac{1}{3})^{\frac{n-1}{2}-1}x_3 \right| +$

$$\sum_{\substack{n=6 \\ n\, even}}^{\infty} \sum_{k=0}^{\frac{n}{2}-3} \left| 3^k (\lambda - \frac{1}{3})^k x_{n-(2k+1)} \right| + \sum_{\substack{n=5 \\ n\, odd}}^{\infty} \sum_{k=0}^{\frac{n-1}{2}-2} \left| 3^k (\lambda - \frac{1}{3})^k x_{n-(2k+1)} \right| \quad (13)$$

This is a geometric series with the common ratio, $r = 3(\lambda - \frac{1}{3})$. This series converge only if $|r| < 1$, that is if $\left|3(\lambda - \frac{1}{3})\right| = 3\left|\lambda - \frac{1}{3}\right| < 1$ or $\left|\lambda - \frac{1}{3}\right| < \frac{1}{3}$. □

2.3 The spectrum of A operator on c

Corollary 2.6. *For matrix A, we have*

$$A - I\lambda = \begin{pmatrix} 1-\lambda & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2}-\lambda & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda & 0 & 0 & \cdots \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda & \cdots \\ & & \cdots & & & \end{pmatrix} \quad (14)$$

$$M = (A - I\lambda)^{-1} is\, given\, by\ m_{nk} = \begin{cases} \frac{1}{a_{nn}}, & n = k \\ \frac{(-1)^{n-k}}{\prod_{j=k}^{n} a_{nn}} D_{n-k}^{(k)}, & (0 \le k \le n-1),\ (n, k \in \mathbb{N}_0) \\ 0, & (k > n) \end{cases} \quad (15)$$

[9] where

$$D_{n-k}^{(k)} = \begin{vmatrix} a_{1k} & a_{1k+1} & 0 & 0 & \cdots 0 \\ a_{2k} & a_{2k+1} & a_{2k+2} & 0 & \cdots 0 \\ 0 & a_{3k+1} & a_{3k+2} & a_{3k+3} & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \vdots & \ddots & \ddots & \ddots\, a_{n,n+k-1} \end{vmatrix} \quad (16)$$

for k=0, $D_n^{(0)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2}-\lambda & 0 & 0 & \cdots & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda & 0 & \cdots & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda & \cdots & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{3} & \frac{1}{3} \end{vmatrix}$, *which is an* $n \times n$ *tridiagonal matrix.*

for $k \geq 1$, $D_{n-k}^{(k)} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3}-\lambda & 0 & \cdots & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda & \cdots & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \frac{1}{3} & \frac{1}{3} \end{vmatrix}$, *this is an* $n-k \times n-k$ *tridiagonal matrix.*

Substituting in equation 15 gives matrix M as ,

$$M = \begin{pmatrix} \frac{1}{1-\lambda} & 0 & 0 & 0 & 0 & \cdots \\ \frac{-1}{2(1-\lambda)(\frac{1}{2}-\lambda)} & \frac{1}{\frac{1}{2}-\lambda} & 0 & 0 & 0 & \cdots \\ \frac{\{\frac{1}{2}-(\frac{1}{2}-\lambda)\}}{3(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)} & \frac{-1}{3(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)} & \frac{1}{\frac{1}{3}-\lambda} & 0 & 0 & \cdots \\ \frac{-\{\frac{1}{2}(1-(1-3\lambda))-(\frac{1}{2}-\lambda)}{3^2(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^2} & \frac{(1-(1-3\lambda)}{3^2(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^2} & \frac{-1}{3^2(\frac{1}{3}-\lambda)^2} & \frac{1}{\frac{1}{3}-\lambda} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix} \tag{17}$$

that is

for $k=0$, $m_{00} = \frac{1}{1-\lambda}$, $m_{n0} = \frac{(-1)^n}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}} D_n^{(0)} = \frac{(-1)^n(\frac{1}{3}D_{n-1}^{(0)} - \frac{1}{3}(\frac{1}{3}-\lambda)D_{n-2}^{0})}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}$,

for $k=1$, $m_{11} = \frac{1}{\frac{1}{2}-\lambda}$, $m_{n1} = \frac{(-1)^{n-1}}{(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}} D_{n-1}^{(1)} = \frac{(-1)^{n-1}(\frac{1}{3}D_{n-2}^{(1)} - \frac{1}{3}(\frac{1}{3}-\lambda)D_{n-3}^{(1)})}{(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}$
for

$$k \geq 2, m_{nn} = \frac{1}{\frac{1}{3}-\lambda}, m_{nk} = \frac{(-1)^{n-k}}{(\frac{1}{3}-\lambda)^{n-k+1}} D_{n-k}^{(k)} = \frac{(-1)^{n-k}(\frac{1}{3}D_{n-k-1}^{(k)} - \frac{1}{3}(\frac{1}{3}-\lambda)D_{n-k-2}^{(k)})}{(\frac{1}{3}-\lambda)^{n-k+1}} \tag{18}$$

Direct computations shows that $(A - I\lambda)M = M(A - I\lambda) = I$, hence $M = (A - I\lambda)^{-1}$

Theorem 2.7. *The spectrum* $\sigma(A)$ *of* $A \in B(c)$ is the set $\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{3}\right| \leq \frac{1}{3}\} \cup \{1\}$

Proof. We show that $(A - I\lambda)^{-1} \in B(c)$ for all $\lambda \in \mathbb{C}$ such that $\left|\lambda - \frac{1}{3}\right| > \frac{1}{3}$
for $k = 0$, $D_1^{(0)} = \frac{1}{2}$
$D_2^{(0)} = \frac{1}{2}(\frac{1}{3}) - (\frac{1}{2}-\lambda)\frac{1}{3} = \frac{1}{3}\{\frac{1}{2} - (\frac{1}{2}-\lambda)\}$
$D_3^{(0)} = \frac{1}{3^2}\{\frac{1}{2} - \frac{1}{2}(1-3\lambda) - (\frac{1}{2}-\lambda)\} = \frac{1}{3^2}\{\frac{1}{2}(1-(1-3\lambda)) - (\frac{1}{2}-\lambda)\}$
$D_4^{(0)} = \frac{1}{3^3}\{\frac{1}{2} - \frac{2}{2}(1-3\lambda) - (\frac{1}{2}-\lambda) - (\frac{1}{2}-\lambda)(1-3\lambda)\} = \frac{1}{3^3}\{\frac{1}{2}(1-2(1-3\lambda)) - (\frac{1}{2}-\lambda)(1-(1-3\lambda))\}$
$D_5^{(0)} = \frac{1}{3^4}\{\frac{1}{2} - \frac{3}{2}(1-3\lambda) + \frac{1}{2}(1-3\lambda)^2 - (\frac{1}{2}-\lambda) + 2(\frac{1}{2}-\lambda)(1-3\lambda)\} = \frac{1}{3^4}\{\frac{1}{2}(1 - 3(1-3\lambda) + (1-3\lambda)^2) - (\frac{1}{2}-\lambda)(1+2(1-3\lambda)\}$

$D_n^{(0)} = \frac{1}{3^{n-1}}\left\{\frac{1}{2}\left(\sum_{k=0}^{\frac{n}{2}-1} a_k(1-3\lambda)^k\right) - (\frac{1}{2}-\lambda)\left(\sum_{k=0}^{\frac{n}{2}-1} b_k(1-3\lambda)^k\right)\right\}$ when n is even, and

$$D_n^{(0)} = \frac{1}{3^{n-1}}\left\{\frac{1}{2}\left(\sum_{k=0}^{\frac{n-1}{2}} a_k(1-3\lambda)^k\right) - (\frac{1}{2}-\lambda)\left(\sum_{k=0}^{\frac{n-1}{2}} b_{k-1}(1-3\lambda)^{k-1}\right)\right\} \tag{19}$$

when n is odd, where $a_{k's}$ and $b_{k's}$ are integers.

Substituting equation 15 gives the n^{th} entry as

$m_{n0} = \frac{(-1)^n}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}} D_n^{(0)} = \frac{(-1)^n\left\{\frac{1}{2}\left(\sum_{k=0}^{\frac{n}{2}-1} a_k(1-3\lambda)^k\right) - (\frac{1}{2}-\lambda)\left(\sum_{k=0}^{\frac{n}{2}-1} b_k(1-3\lambda)^k\right)\right\}}{3^{n-1}(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}$,when n is even, and

$m_{n0} = \frac{(-1)^n}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}} D_n^{(0)} = \frac{(-1)^n\left\{\frac{1}{2}\left(\sum_{k=0}^{\frac{n-1}{2}} a_k(1-3\lambda)^k\right) - (\frac{1}{2}-\lambda)\left(\sum_{k=0}^{\frac{n-1}{2}} b_{k-1}(1-3\lambda)^{k-1}\right)\right\}}{3^{n-1}(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}$,when n is odd

as $n \to \infty$, the colums $m_{no} \to 0$ only if the denominator tends to $\infty$and the denominator tends to ∞ provided $\left|3(\frac{1}{3}-\lambda)\right| > 1$.

Similarly for $k \geq 1$,the denominator tends to ∞ provided $\left|3(\frac{1}{3}-\lambda)\right| > 1$ or $\left|\frac{1}{3}-\lambda\right| > \frac{1}{3}$ □

Which proves theorem 1.1 (i). Summing the entries of the matrix 17 along the n^{th} row
$\sum_{k=0}^{\infty} |m_{nk}| = \left|\frac{(-1)^n D_n^{(0)}}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}\right| + \left|\frac{(-1)^{n-1} D_{n-1}^{(1)}}{(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}\right| + \sum_{k=2}^{n}\left|\frac{(-1)^{n-k}(\frac{1}{3}D_{n-k-1}^{(k)} - \frac{1}{3}(\frac{1}{3}-\lambda)D_{n-k-2}^{(k)})}{(\frac{1}{3}-\lambda)^{n-k+1}}\right| =$ s_n say for $n \geq 0$ $\sup_n\{s_n\} \leq K < \infty$, provided $\lambda \in \mathbb{C}$ such that $\left|\frac{1}{3}-\lambda\right| > \frac{1}{3}$, hence satisfies part (iii). For part (ii), we have $M = (A - I\lambda)^{-1}$ and $(A - I\lambda)(A - I\lambda)^{-1} = I$. Now $M\delta = \sum_{k=0}^{n} m_{nk}$, where $\delta = (1, 1, , 1\cdots)^T$. Also $(A - I\lambda)^{-1}(A - I\lambda) = M(A - I\lambda) = I$, multiplying by δ on both sides $M(A - I\lambda)\delta = I\delta$. Since $A\delta = \delta$, we have $M(\delta - \lambda\delta) = \delta$ or $M(1-\lambda)\delta = \delta$. Therefore

$$M\delta = \frac{1}{1-\lambda}\delta \tag{20}$$

That is

$$\sum_{k=0}^{n} m_{nk} = \frac{1}{1-\lambda} \tag{21}$$

hence $\lim_n \sum_{k=0}^{\infty} m_{nk} = \lim_n \frac{1}{1-\lambda} = \frac{1}{1-\lambda} < \infty$ provided $\lambda \in \mathbb{C}$ such that $\lambda \neq 1$.

Therefore $(A - I\lambda)^{-1} \in B(c)$ if $\lambda \in \mathbb{C}$ such that $\left|\frac{1}{3} - \lambda\right| > \frac{1}{3}$. Which implies $(A-I\lambda)^{-1} \notin B(c)$ if $\lambda \in \mathbb{C}$ such that $\left|\frac{1}{3} - \lambda\right| \leq \frac{1}{3}$. Clearly, when $\lambda = 1$, column 0 is infinite therefore the inverse does not exist. Hence $\sigma(A) =\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{3}\right| \leq \frac{1}{3}\} \cup \{1\}$.

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