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Portfolio optimization for an insider under partial information

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ABSTRACT

In this article, we seek to solve the problem of stochastic filtering of the unobserved drift of the stock price in the presence of privileged information. Working within a finite time investment horizon, the privileged information which is a function of the future value of the stock price, is modeled such that its quality improves as we move towards the information reveal date. The hidden/unobserved drift is modeled as a Gaussian process. Combining the techniques of progressive enlargement of filtration and stochastic filtering of linear state-space models, we obtain explicit analytic results for the insider's estimates of the unobserved drift process. In addition, we obtain the optimal portfolio strategy for an insider having the log utility function. Our numerical results reveal that when the quality of privileged information is high, the insider would require less initial capital as compared to the regular trader who has no access to the privileged information. Further, we show how the stock price volatility influences the value of the insider's privileged information, with period of high volatility pointing to increased value of the privileged information.

Introduction

Information asymmetry in the financial markets exists since different classes of investors have access to varying sets of information. Information asymmetry between different investors may exist by the very fact that some investors (the insider) may be more informed on the future value of a particular stock compared to the rest of the market (the regular trader). In this work, we consider a financial market constituting of one risk-free asset (bond) and one risky asset (stock). The drift of the stock price is unobserved by all classes of traders. However, one class of traders (the insider) has access to privileged information concerning the future value of the stock. The insider's privileged information on the future value of the stock price is deformed by some noisy process. As the terminal time of investment nears, the noise reduces and the insider's privileged information gets clearer. Thus, this work fuses partial information scenario and dynamic enlargement of filtration.

We begin with a linear state-space model whereby the observed process Y_t is a function of the stock price S_t. The insider does not observe the process X_t driving the drift of the stock price. The process X_t has been modeled as a Gaussian process. The investor's filtration $\mathcal{Y}_t$ is enlarged by the filtration $\sigma(Z_s, s \le t)$ whereby $Z_t = Y_T + \varepsilon M_{(T-t)^\theta}$ such that Y_T is a function of the future value of the stock price and this has been deformed by the independent noise process $M_{(T-t)^\theta}$. As time

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evolves, the noise clears and at terminal time T, the insider will have access to perfect information. The impact of filtration enlargement is that the insider's risk premium will have an additional term β_t known as the information drift, which is not necessarily adapted to the enlarged observation filtration, denoted by $\tilde{\mathcal{Y}}_t$. Stochastic filtering techniques are applied to infer the hidden drift process X_t and the information drift β_t, and this converts the partial and insider information model to a full information scenario. Note that on account of filtration enlargement, $\tilde{\mathcal{Y}}_0$ is not a trivial σ-algebra. The financial model is applied to solve a utility maximization problem for an insider having the log utility function. We determine the insider's optimal portfolio strategy and compare the results with the case of a regular trader having no access to the privileged information.

In mathematical finance, the insider's additional information is modeled using techniques of enlargement of filtration. Initial enlargement of filtration applies when the insider's privileged information is modeled as a random variable. Some previous application of initial enlargement of filtration to optimal investment include [1] and [2]. In [1], the expected terminal wealth is unbounded for an insider who is perfectly informed on the future value of the stock price and has the log utility function. [2] computed the insider's additional expected utility whereby the privileged information consisted of the future value of the stock price perturbed by an independent noise which is constant throughout the time interval. An excellent study of filtration enlargement in an incomplete market with jumps is [3], where the authors were able to obtain the optimal portfolio strategy for an insider whose additional information is the actual number of Poisson jumps within a finite time interval. On the other hand, whenever the insider's privileged information is modeled as a stochastic process, then we can apply the progressive enlargement of filtration techniques, see e.g. [4] where it was proven that if the rate at which the deforming noise in the privileged information disappears is slow enough, then the insider will have finite additional utility. On the other hand, [5] obtained dynamic filtration enlargement results by applying the theory of weak convergence of σ-algebras. In [6], the authors solved a risk-sensitive portfolio optimization problem for a insider who is also a large trader, such that the insider's portfolio holding has an impact on the price of the risky asset. In [7], the optimal portfolio strategies and terminal wealth were obtained for different insiders having access to varying sets of privileged information: initial "strong" information on the stock price corresponding to initial enlargement of filtration, progressive "strong" information which gets clearer as time evolves corresponding to dynamic enlargement of filtration and lastly the insider with "weak" additional information pertaining to the probability law of the future value of the stock price. The study revealed that an insider having access to a more relevant additional information will end up having higher expected terminal wealth, the "best" information being the initial "strong" information. However, it is worth noting that the concept of "weak" insider information as introduced by [8] entails a change of probability measure rather than filtration enlargement.

The objective of stochastic filtering is to find the best estimate of a hidden process X_t that is only partially seen through the observation process Y_t. Excellent textbook treatment of stochastic filtering include [9] and [10]. Some studies on partial information and investment models include [11], whereby the authors considered a financial market where the drift of the stock price is driven by a hidden continuous time Markov chain process. For the related case of an unobserved drift modeled as a Gaussian process, [12] solved an optimal portfolio problem for an investor using several utility functions. An optimal portfolio problem in a partially observed stochastic volatility model was solved by [13]. In [14], the authors studied a market model whereby the unobserved drift process (modeled as an Ornstein-Uhlenbeck process) is filtered conditioned on a continuous time stock return observations and discrete time expert opinion. [15] considered a market model whereby the unobserved drift parameter is estimated on an observation filtration initially enlarged with the future value of the stock price. The future value of the stock price was deformed by an independent constant-variance noise term, hence this work combined initial enlargement of filtration with stochastic filtering. More application of partial information models in other areas include [16] who considered stochastic filtering problem whereby the insider's filtration is enlarged by perfect knowledge of the future value of the observed process, with an application to the linear regulator problem. In a different mathematical application, [17] solved a continuous time optimal credit limit problem for a lender who has partial information on the borrower's credit quality. The lender receives additional information from the borrower's social network.

Our work is closest to [15] and [16] with regards to combining enlargement of filtration and stochastic filtering techniques. However in [15], the additional insider information was modeled as a function of the stock price deformed by a constant-variance noise term, i.e. the quality of insider information does not improve as time evolves. Similarly, in [16] the insider is assumed to have perfect knowledge of the future value of the observed process. The key contribution of our work is that we combine the dynamic enlargement of filtration of [4] and stochastic filtering of linear state-space models. Thus, we consider a case whereby inasmuch as the insider's privileged information gets better, the insider is still faced with an additional challenge of not observing the drift of the stock price. Upon enlarging the filtration and estimating the unobserved drift process, we analytically obtain the explicit optimal portfolio strategy for an insider having the log utility function. In our numerical computations, we show that in the extreme case whereby the insider has access to accurate information of the future value of the stock price, there will be an arbitrage opportunity for the insider as found e.g. in [1]. In addition, we show that as the quality of insider's additional information deteriorates, the regular trader would require less amounts of extra initial capital inorder to match the insider's expected terminal wealth. Further, the numerical results reveal that higher levels of stock price volatility points to increased value of the insider's privileged information, and thus the insider would need less amounts of initial capital to match the regular trader's expected terminal wealth.

The paper is organized as follows. In Section 2, we present our financial market model. Our main results on enlargement of filtration and stochastic filtering are presented in Section 3. Section 4 presents the results on optimal portfolio control for both the insider and the regular trader. Some brief numerical computations are also presented in this section. Section 5 concludes.

The financial market model

Let $(\Omega, \mathcal{A}, \mathbb{A}, \mathbb{P})$ with $\mathbb{A} = (\mathcal{A}_t)_{t\geq 0}$ be a filtered probability space satisfying the usual conditions of right continuity and completeness. All processes are assumed to be $\mathbb{A}$ adapted. Working on the finite time interval $[0, T]$, consider a continuous time financial market consiting of a risk-free asset (bond) and a risky asset (stock). The stock price S_t has the dynamics

$$dS_t = S_t(\alpha X_t dt + \delta dB_t)$$

and the price of the risk-free bond is modeled as $dS_t^0 = rS_t^0 dt$. Without loss of generality, we assume that the risk-free interest rate $r = 0$ so that $S_t^0 = 1$ for all $t \geq 0$. The drift X_t of the stock price is a stochastic process, unobserved by the investor. The investor only observes the process Y_t and we have the following linear state-space model

$$\begin{aligned} dX_t &= \sigma dW_t,\, X_0 = x_0 \\ dY_t &= \frac{dS_t}{S_t} = \alpha X_t dt + \delta dB_t \end{aligned} \tag{1}$$

where W_t and B_t are independent Brownian motion processes, $\alpha \in \mathbb{R}$, and $\delta, \sigma > 0$ are constants. We define $\mathcal{F}_t = \sigma(W_s, B_s, s \leq t)$ for $0 \leq t \leq T$ to be the natural filtration generated by the Brownian motion processes. $x_0 \sim \mathcal{N}(\mu_0, \varsigma_0)$ is a Gaussian random variable with the given mean and variance, and is independent of W and B. Thus x_0 is $\mathcal{F}_0$ measurable. The unobserved state process X_t is modeled as a Gaussian process, whilst the observation process Y_t is a linear function of the state process. Note that the investor's observation filtration is $\mathcal{Y}_t = \sigma(Y_s, s \leq t) = \sigma(S_s, s \leq t)$. Thus, the investor's observation filtration is the history of the stock price. Knowledge of Y_T at time $t < T$ is akin to having information on the future price of the stock. Define the process

$$Z_t = Y_T + \varepsilon M_{(T-t)^\theta} \tag{2}$$

where $M_{(T-t)^\theta}$ is a Brownian motion process independent of $\mathcal{F}_T$, $0 < \theta < 1$ and $\varepsilon > 0$ is the noise parameter. We define the following filtrations;

$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(Z_s, s \leq t)$ is the progressively enlarged filtration containing $\mathcal{F}_t$ to which Z is adapted.

$\tilde{\mathcal{Y}}_t = \mathcal{Y}_t \vee \sigma(Z_s, s \leq t)$ is the progressively enlarged observation filtration. This is the insider's observation filtration.

$\bar{\mathcal{F}}_t = \mathcal{F}_t \vee \sigma(Y_T)$ is the initially enlarged filtration containing $\mathcal{F}_t$.

$\mathcal{H}_t = \bar{\mathcal{F}}_t \vee \sigma(M_{(T-t)^\theta}, s \leq t)$ is the progressively enlarged filtration containing $\bar{\mathcal{F}}_t$ to which $M_{(T-t)^\theta}$ is adapted.

For $0 \leq s \leq t$ we have the representation

$$\begin{aligned} Y_t &= \alpha t x_0 + \alpha\sigma \int_0^t W_u du + \delta B_t \\ &= Y_s + \alpha(t-s)X_s + \alpha\sigma \int_s^t (W_u - W_s) du + \delta(B_t - B_s) \end{aligned} \tag{3}$$

This means that conditioned on $\mathcal{F}_s$, Y_t has a Gaussian distribution with mean $Y_s + \alpha(t-s)X_s$. It is easy to show that the conditional variance of Y_t given $\mathcal{F}_s$ is given as

$$\mathbb{E}\left(\alpha\sigma \int_s^t (W_u - W_s) ds + \delta(B_t - B_s) | \mathcal{F}_s\right)^2 = \alpha^2\sigma^2 \frac{(t-s)^3}{3} + \delta^2(t-s)$$

Thus, denoting the $\mathcal{F}_t$ conditional distribution of Y_T as π_{T-t} we have

$$\pi_{T-t}(t, X_t, Y_t; \nu) = \frac{1}{\sqrt{2\pi\left(\alpha^2\sigma^2 \frac{(T-t)^3}{3} + \delta^2(T-t)\right)}} \exp\left\{ -\frac{(\nu - (Y_t + \alpha(T-t)X_t))^2}{2\left(\alpha^2\sigma^2 \frac{(T-t)^3}{3} + \delta^2(T-t)\right)} \right\} \tag{4}$$

Enlargement of filtration and stochastic filtering

Enlargement of filtration

Let $\mathbb{E}(Y_T|\mathcal{F}_t) = m_t$ and $Var(Y_T|\mathcal{F}_t) = \vartheta_t$. Given the $\mathcal{F}_t$ conditional density of Y_T as π_{T-t}, we decompose the conditional density as

$$\pi_{T-t}(\upsilon) = \underbrace{\frac{1}{\sqrt{2\pi\vartheta_0}} e^{-\frac{1}{2}\left\{\frac{(\upsilon-m_0)^2}{\vartheta_0}\right\}}}_{\pi_T(\upsilon)} \underbrace{\sqrt{\frac{\vartheta_0}{\vartheta_t}} e^{-\frac{1}{2}\left\{\frac{(\upsilon-m_t)^2}{\vartheta_t} - \frac{(\upsilon-m_0)^2}{\vartheta_0}\right\}}}_{M^\upsilon(t,X_t,Y_t)} \tag{5}$$

where $\pi_T(\upsilon)$ is the Gaussian density of Y_T with mean $m_0 = \alpha\mu_0$ and variance $\vartheta_0 = \alpha^2\varsigma_0 + \frac{\alpha^2\sigma^2}{3} + \delta^2$.

Lemma 3.1. *$M^\upsilon(t, X_t, Y_t)$ is a $\mathcal{F}_t$ martingale.*

Proof. The proof is attained by use of Ito's formula. We first obtain the partial derivatives of $M^\upsilon(t, X_t, Y_t)$ w.r.t t, X_t and Y_t as follows: the partial derivatives w.r.t t is

$$\begin{aligned}\frac{\partial M^\upsilon(t,X_t,Y_t)}{\partial t} =& \frac{1}{2}\frac{M^\upsilon(t,X_t,Y_t)}{\vartheta_t}\left(\alpha^2\sigma^2(T-t)^2+\delta^2\right) - \alpha X_t M^\upsilon(t,X_t,Y_t)\frac{(\upsilon-m_t)}{\vartheta_t}\\ &-\frac{1}{2}M^\upsilon(t,X_t,Y_t)\frac{(\upsilon-m_t)^2}{\vartheta_t^2}\left(\alpha^2\sigma^2(T-t)^2+\delta^2\right)\end{aligned} \tag{6}$$

The first and second partial derivatives w.r.t X_t are respectively

$$\frac{\partial M^\upsilon(t,X_t,Y_t)}{\partial x} = \alpha(T-t)M^\upsilon(t,X_t,Y_t)\frac{(\upsilon-m_t)}{\vartheta_t}$$

and

$$\frac{\partial^2 M^\upsilon(t,X_t,Y_t)}{\partial x^2} = \alpha^2(T-t)^2 M^\upsilon(t,X_t,Y_t)\frac{(\upsilon-m_t)^2}{\vartheta_t^2} - \alpha^2(T-t)^2\frac{M^\upsilon(t,X_t,Y_t)}{\vartheta_t}$$

Finally, the first and second partial derivatives w.r.t Y_t are respectively

$$\frac{\partial M^\upsilon(t,X_t,Y_t)}{\partial y} = M^\upsilon(t,X_t,Y_t)\frac{(\upsilon-m_t)}{\vartheta_t}$$

and

$$\frac{\partial^2 M^\upsilon(t,X_t,Y_t)}{\partial y^2} = M^\upsilon(t,X_t,Y_t)\frac{(\upsilon-m_t)}{\vartheta_t}$$

Proceeding to apply Ito's formula, we get that

$$\begin{aligned} dM^\upsilon(t,X_t,Y_t) &= \alpha\sigma(T-t)M^\upsilon(t,X_t,Y_t)\tfrac{(\upsilon-m_t)}{\vartheta_t}dW_t + \delta M^\upsilon(t,X_t,Y_t)\tfrac{(\upsilon-m_t)}{\vartheta_t}dB_t\\ &= \psi_t^1 M^\upsilon(t,X_t,Y_t)dW_t + \psi_t^2 M^\upsilon(t,X_t,Y_t)dB_t\end{aligned} \tag{7}$$

Thus $M^\upsilon(t, X_t, Y_t)$ is a $\mathcal{F}_t$ local martingale. Note that

$$\begin{aligned}\int_0^T \mathbb{E}\left(\psi_t^1 M^\upsilon(t,X_t,Y_t)\right)^2 dt &= \int_0^T \mathbb{E}\left(\alpha^2\sigma^2(T-t)^2\tfrac{(\upsilon-m_t)^2}{\vartheta_t^2}\tfrac{\vartheta_0}{\vartheta_t}e^{-\left\{\frac{(\upsilon-m_t)^2}{\vartheta_t}-\frac{(\upsilon-m_0)^2}{\vartheta_0}\right\}}\right)dt\\ &\le e^{\frac{(\upsilon-m_0)^2}{\vartheta_0}}\int_0^T \tfrac{\vartheta_0}{\vartheta_t}\mathbb{E}\left(\alpha\sigma(T-t)\tfrac{(\upsilon-m_t)}{\vartheta_t}\right)^2 dt < \infty\end{aligned} \tag{8}$$

The finite expectation of the integrand is as a result of the Gaussian property of X_t and Y_t, thus the respective second moments exist. Similar arguments can be applied to show that $\int_0^T \mathbb{E}(\psi_t^2 M^\upsilon(t,X_t,Y_t))^2 dt < \infty$ Thus $M^\upsilon(t, X_t, Y_t)$ is a true $\mathcal{F}_t$ martingale. □

Lemma 3.2. *W_t and B_t are semimartingales in the initially enlarged filtration $\bar{\mathcal{F}}_t$ with their decompositions given as*

$$W_t = \bar{W}_t + \int_0^t \psi_s^1 ds \tag{9}$$

$$B_t = \bar{B}_t + \int_0^t \psi_s^2 ds \tag{10}$$

where $\bar{W}_t$ and $\bar{B}_t$ are $\bar{\mathcal{F}}_t$ Brownian motion processes.

Proof. We do obtain the semimartingale decomposion of W_t in the initially enlarged filtration $\bar{\mathcal{F}}_t$ by applying Jacod's theorem (see e.g. [18]). Since the conditional distribution of Y_T given $\mathcal{F}_t$ is $\pi_{T-t}(\upsilon)$, the semimartingale decomposion of W_t is obtained as

$$\bar{W}_t = W_t - \int_0^t \frac{\langle \pi_{T-.} W_. \rangle_s}{\pi_{T-s}} = W_t - \int_0^t \psi_s^1 ds$$

Let $Z_s = 1_{F\cap A}$ be a $\bar{\mathcal{F}}_s$ measurable random variable where 1 is the indicator function, F is a $\mathcal{F}_s$-measurable set and A is measurable w.r.t $\sigma(Y_T)$ i.e. there exists a Borel set B such that $A = [Y_T \in B]$. To show that 9 holds, we need only prove that

$$\mathbb{E}\left(Z_s\left(\bar{W}_t - \bar{W}_s\right)\right) = 0 \tag{11}$$

Applying the tower property of conditional expectation, we get

$$\mathbb{E}\left(Z_s\left(\bar{W}_t - \bar{W}_s\right)\right) = \mathbb{E}\left(1_{F\cap A}\left(W_t - W_s - \int_s^t \psi_u^1 du\right)\right) \tag{12}$$
$$= \mathbb{E}(1_F(W_t - W_s)\mathbb{E}(1_A|F_t)) - \mathbb{E}\left(1_A 1_F \int_s^t \psi_u^1 du\right)$$

Since $\mathbb{E}(1_A|F_t) = \mathbb{P}(A|F_t) = \mathbb{P}([Y_T \in B]|F_t)$, then

$$\mathbb{E}\left(Z_s\left(\bar{W}_t - \bar{W}_s\right)\right) = \mathbb{E}\left(1_F(W_t - W_s)\int_B M^\upsilon(t, X_t, Y_t)\pi_T(\upsilon)d\upsilon\right) - \mathbb{E}\left(1_A 1_F \int_s^t \psi_u^1 du\right)$$
$$= \mathbb{E}\left(1_F \int_B \left(W_t M^\upsilon(t, X_t, Y_t) - W_s M^\upsilon(s, X_s, Y_s)\right)\pi_T(\upsilon)d\upsilon\right) - \mathbb{E}\left(1_A 1_F \int_s^t \psi_u^1 du\right) \tag{13}$$

By Ito's formula, we have

$$d\left(W_t M^\upsilon(t, X_t, Y_t)\right) = W_t dM^\upsilon(t, X_t, Y_t) + M^\upsilon(t, X_t, Y_t)dW_t + d < W, M^\upsilon(., X_., Y_.)>_t$$
$$= W_t M^\upsilon(t, X_t, Y_t)\left(\psi_t^1 dW_t + \psi_t^2 dB_t\right) + M^\upsilon(t, X_t, Y_t)dW_t + \psi_t^1 M^\upsilon(t, X_t, Y_t)dt \tag{14}$$

Applying Fubini's theorem, the tower property of conditional expectation and 14, we get

$$\mathbb{E}\left(Z_s\left(\bar{W}_t - \bar{W}_s\right)\right) = \int_B \mathbb{E}[1_F \int_s^t \psi_u^1 M^\upsilon(u, X_u, Y_u)du]\pi_T(\upsilon)d\upsilon - \mathbb{E}\left(1_A 1_F \int_s^t \psi_u^1 du\right) = 0 \tag{15}$$

Eq. 11 applies to all bounded real valued $\bar{\mathcal{F}}_s$ measurable functions by way of the Monotone class argument. Thus 9 holds. Similarily, it can be shown that $\bar{B}_t$ is a $\bar{\mathcal{F}}_s$ martingale. Levy characterization theorem can then be applied to show that indeed $\bar{W}_t$ and $\bar{B}_t$ are $\overline{\mathcal{F}}_t$ Brownian motion processes. □

Proposition 3.3. *W_t and B_t are semimartingales in the progressively enlarged filtration $\mathcal{G}$ with their respective decomposition being,*

$$W_t = \tilde{W}_t + \int_0^t \alpha\sigma(T-s)\frac{(Z_s - (Y_s + \alpha(T-s)X_s))}{\left(\alpha^2\sigma^2\frac{(T-s)^3}{3} + \delta^2(T-s) + \varepsilon^2(T-s)^\theta\right)}ds$$
$$B_t = \tilde{B}_t + \int_0^t \delta\frac{(Z_s - (Y_s + \alpha(T-s)X_s))}{\left(\alpha^2\sigma^2\frac{(T-s)^3}{3} + \delta^2(T-s) + \varepsilon^2(T-s)^\theta\right)}ds \tag{16}$$

with $\tilde{W}_t$ and $\tilde{B}_t$ being $\mathcal{G}_t$ Brownian motion processes.

Proof. To prove 16, we note that $Z_t = Y_T + \varepsilon M_{(T-t)^\theta}$ conditioned on $\mathcal{F}_t$ has a Gaussian distribution with mean $\mathbb{E}(Z_t|\mathcal{F}_t) = m_t$ and $Var(Z_t|\mathcal{F}_t) = \vartheta_t + \varepsilon^2(T-t)^\theta$. From lemma 3.2, we know that W_t and B_t are semimartingales in the initially enlarged filtration $\bar{\mathcal{F}}_t$. Using proposition 4 of [4] together with the properties of Gaussian random variables, we obtain the $\mathcal{G}_t$ optional projections of ψ_s^1 and ψ_s^2 as

$$\gamma_s = \mathbb{E}(\psi_s^1|\mathcal{F}_s \vee \sigma(Z_s)) = \alpha\sigma(T-s)\frac{(Z_s - (Y_s + \alpha(T-s)X_s))}{\left(\alpha^2\sigma^2\frac{(T-s)^3}{3} + \delta^2(T-s) + \varepsilon^2(T-s)^\theta\right)}$$

$$\beta_s = \mathbb{E}(\psi_s^2|\mathcal{F}_s \vee \sigma(Z_s)) = \delta \frac{(Z_s - (Y_s + \alpha(T-s)X_s))}{\left(\alpha^2\sigma^2\frac{(T-s)^3}{3} + \delta^2(T-s) + \varepsilon^2(T-s)^\theta\right)} \tag{17}$$

Thus the decomposition of W and B in the progressively enlarged filtration $\mathcal{G}_t$ are

$$W_t = \tilde{W}_t + \int_0^t \alpha\sigma(T-s)\frac{(Z_s - (Y_s + \alpha(T-s)X_s))}{\left(\alpha^2\sigma^2\frac{(T-s)^3}{3} + \delta^2(T-s) + \varepsilon^2(T-s)^\theta\right)}ds$$

$$B_t = \tilde{B}_t + \int_0^t \delta\frac{(Z_s - (Y_s + \alpha(T-s)X_s))}{\left(\alpha^2\sigma^2\frac{(T-s)^3}{3} + \delta^2(T-s) + \varepsilon^2(T-s)^\theta\right)}ds \tag{18}$$

Note that

$$\mathbb{E}(W_t|\mathcal{G}_t) = \mathbb{E}\left(\bar{W}_t + \int_0^t \psi_s^1 ds|\mathcal{G}_t\right) = \mathbb{E}(\bar{W}_t|\mathcal{G}_t) + \int_0^t \mathbb{E}\left(\psi_s^1|\mathcal{G}_t\right)ds$$

such that $\mathbb{E}(\bar{W}_t|G_t) = \tilde{W}_t$. Since $\bar{W}_t$ is a $\bar{\mathcal{F}}_t$ martingale and $\bar{\mathcal{F}}_t \subset \mathcal{H}_t$ then it means that $\bar{W}_t$ is also a $\mathcal{H}_t$ martingale. Noting that $\mathcal{G}_t \subset \mathcal{H}_t$, the projection of $\bar{W}_t$ on $\mathcal{G}_t$ will be a $\mathcal{G}_t$ martingale. Indeed

$$\mathbb{E}(\tilde{W}_t|\mathcal{G}_s) = \mathbb{E}(\mathbb{E}(\bar{W}_t|\mathcal{G}_t)|\mathcal{G}_s) = \mathbb{E}(\bar{W}_t|\mathcal{G}_s) = \mathbb{E}(\mathbb{E}(\bar{W}_t|\mathcal{H}_s)|\mathcal{G}_s) = \mathbb{E}(\bar{W}_s|\mathcal{G}_s) = \tilde{W}_s$$

A similar argument can be invoked to show that $\tilde{B}_t$ is a $\mathcal{G}_t$ martingale. By applying the Levy characterization theorem, we conclude the proof that $\tilde{W}_t$ and $\tilde{B}_t$ are indeed Brownian motion processes in the progressively enlarged filtration $\mathcal{G}_t$ □

Define

$$\Sigma_t = \alpha^2\sigma^2\frac{(T-t)^3}{3} + \delta^2(T-t) + \varepsilon^2(T-t)^\theta$$

and the information drift process

$$\beta_t = \frac{\delta(Z_t - (Y_t + \alpha(T-t)X_t))}{\Sigma_t}$$

We note that the information drift process β_t is neither adapted to the observation filtration $\mathcal{Y}_t$ nor the enlarged observation filtration $\tilde{\mathcal{Y}}_t$ since one of the terms contains X_t which is unobserved by the insider. The state-observation system of equations in the progressively enlarged filtration $\mathcal{G}_t$ can now be represented as

$$dX_t = \alpha\sigma^2(T-t)\left(\frac{Z_t - (Y_t + \alpha(T-t)X_t)}{\Sigma_t}\right)dt + \sigma d\tilde{W}_t,\ X_0 = x_0$$

$$dY_t = \left(\alpha X_t + \delta^2\frac{(Z_t - (Y_t + \alpha(T-t)X_t))}{\Sigma_t}\right)dt + \delta d\tilde{B}_t \tag{19}$$

The equations are linear on $\mathcal{G}_t$, meaning that the usual Kalman-Bucy filtering technique (see e.g. [9]) can be applied to obtain filtered estimates. However, our stochastic filtering problem is unique in that the observation filtration $\tilde{\mathcal{Y}}_t$ has been enlarged by the process Z_t. Note that $\tilde{\mathcal{Y}}_0$ is not a trivial σ-algebra since it contains Z_0

Stochastic filtering

In this subsection, we infer estimates of the unobserved drift process X_t in 19, conditioned on the insider's observation filtration $\tilde{\mathcal{Y}}_t$. The optimal estimate for X_t in the mean-square sense is the filtered estimate $\mathbb{E}\left(X_t|\tilde{\mathcal{Y}}_t\right)$

Proposition 3.4. *The dynamics of the conditional mean* $\tilde{X}_t = \mathbb{E}(X_t|\tilde{\mathcal{Y}}_t)$ *is given by*

$$d\tilde{X}_t = \alpha\sigma^2(T-t)\left(\frac{(Z_t - Y_t)}{\Sigma_t} - \frac{\alpha(T-t)\tilde{X}_t}{\Sigma_t}\right)dt + \alpha\delta^{-1}\left(1 - \frac{\delta^2(T-t)}{\Sigma_t}\right)P_t dI_t,\ \tilde{X}_0 = \mu \tag{20}$$

with the conditional variance $P_t = \mathbb{E}\left((X_t - \tilde{X}_t)^2|\tilde{\mathcal{Y}}_t\right)$ *having the dynamics*

$$\frac{dP_t}{dt} = -2\alpha^2\sigma^2\frac{\left((T-t)^2\right)}{\Sigma_t}P_t + \sigma^2 - \alpha^2\delta^{-2}\left(1 - \frac{\delta^2(T-t)}{\Sigma_t}\right)^2 P_t^2,\ P_0 = \varsigma \tag{21}$$

Proof. Define I_t as the innovation process whose dynamics is given by the following SDE

$$dI_t = d\tilde{B}_t + \delta^{-1}\left(\alpha(X_t - \tilde{X}_t) - \alpha\delta^2\frac{(T-t)(X_t - \tilde{X}_t)}{\Sigma_t}\right)dt \tag{22}$$

Note that

$$\mathbb{E}(I_t - I_s|\tilde{\mathcal{Y}}_s) = \mathbb{E}\left(\tilde{B}_t - \tilde{B}_s - \int_s^t \delta^{-1}\left(\alpha(X_u - \tilde{X}_u) - \alpha\delta^2\frac{(T-u)(X_u - \tilde{X}_u)}{\Sigma_u}\right)du|\tilde{\mathcal{Y}}_s\right)$$
$$= \mathbb{E}(\tilde{B}_t - \tilde{B}_s) - \int_s^t \delta^{-1}\mathbb{E}\left(\alpha(X_u - \tilde{X}_u) - \alpha\delta^2\frac{(T-u)(X_u - \tilde{X}_u)}{\Sigma_u}|\tilde{\mathcal{Y}}_s\right)du = 0$$

The second equality is by the independent increment property of $\tilde{B}_t$ and Fubini theorem whilst the last equality is due to the tower property of conditional expectation, thus I_t is a $\tilde{\mathcal{Y}}_t$ martingale. That I_t is a $\tilde{\mathcal{Y}}_t$ Brownian motion follows from the Levy characterization theorem. From 19 and applying standard filtering theory results, the dynamics of the conditional mean $\tilde{X}_t = \mathbb{E}(X_t|\tilde{\mathcal{Y}}_t)$ is given by the equation

$$d\tilde{X}_t = \alpha\sigma^2(T-t)\left(\frac{(Z_t - Y_t)}{\Sigma_t} - \frac{\alpha(T-t)\tilde{X}_t}{\Sigma_t}\right)dt + \alpha\delta^{-1}\left(1 - \frac{\delta^2(T-t)}{\Sigma_t}\right)P_t dI_t,\ \tilde{X}_0 = \mu_0$$

where P_t is the conditional variance. $(X_t - \tilde{X}_t)$ is Gaussian and is independent of $\tilde{\mathcal{Y}}_t$, thus

$$P_t = \mathbb{E}\left((X_t - \tilde{X}_t)^2|\tilde{\mathcal{Y}}_t\right) = \mathbb{E}\left((X_t - \tilde{X}_t)^2\right) = \mathbb{E}(X_t^2|\tilde{\mathcal{Y}}_t) - \tilde{X}_t^2 \tag{23}$$

is deterministic. By applying Ito calculus on $\tilde{X}_t^2$ and X_t^2, we conclude that the conditional variance has the dynamics

$$\frac{dP_t}{dt} = -2\alpha^2\sigma^2\frac{(T-t)^2}{\Sigma_t}P_t + \sigma^2 - \alpha^2\delta^{-2}\left(1 - \frac{\delta^2(T-t)}{\Sigma_t}\right)^2 P_t^2,\ P_0 = \varsigma_0$$

□

Corollary 3.5. *When the observation filtration is $\mathcal{Y}_t$, we revert back to the classical case of Kalman-Bucy filtering with the state-observation equation being as is in 1. Similar to the linear filtering results in Proposition 3.4, it can be shown that under $\mathcal{Y}_t$, the filtered estimates and conditional variance have the dynamics*

$$d\hat{X}_t = \frac{\alpha}{\delta}\hat{P}_t d\hat{I}_t,\ \hat{X}_0 = \mu$$
$$\frac{d\hat{P}_t}{dt} = \sigma^2 - \frac{\alpha^2}{\delta^2}\hat{P}_t,\ \hat{P}_0 = \varsigma_0 \tag{24}$$

where $\hat{I}_t$ is the innovation process, $\hat{P}_t$ is the conditional variance and $\hat{X}_t$ the filtered process.

Portfolio optimization

Denote the wealth process of a self-financing portfolio as V_t. Given that $r = 0$ in our financial market model, the wealth process will have the dynamics

$$\frac{dV_t}{V_t} = \pi_t\frac{dS_t}{S_t} = \alpha\pi_t X_t dt + \delta\pi_t dB_t,\ V_0 = v_0$$

where π_t is the proportion of wealth invested in the risky asset at any time t. The admissible portfolio strategies belong to the set

$$\mathcal{A}^{\mathbb{H}} = \left\{\pi = (\pi_t)_{t\in[0,T]}, \pi \text{ is } \mathbb{H}-\text{adapted}, V_t \geqslant 0, t \in [0,T], \mathbb{E}\left(\int_0^T \pi_t^2 dt\right) < \infty\right\}$$

for $\mathbb{H} \in \{\mathcal{Y}, \tilde{\mathcal{Y}}\}$. We consider an insider and regular trader, both having the logarithmic utility function and they seek to maximize their respective expected utility from terminal wealth

$$U^{\mathbb{H}}(v_0) = \sup_{\pi\in\mathcal{A}^{\mathbb{H}}}\mathbb{E}(\log V_T) \tag{25}$$

We are interested in the portfolio strategy $\pi*$ which attains the optimal value in 25

Lemma 4.1. *Let*

$$\tilde{\beta}_t = \mathbb{E}\left(\beta_t|\tilde{\mathcal{Y}}_t\right) = \frac{\delta\left(Z_t - \left(Y_t + \alpha(T-t)\tilde{X}_t\right)\right)}{\Sigma_t}$$

then the following holds

$$\mathbb{E}\left(\alpha\tilde{X}_t + \delta\tilde{\beta}_t\right)^2 < \infty \tag{26}$$

Proof.

$$\begin{aligned}
\mathbb{E}\Big(\alpha\tilde{X}_t+\delta\tilde{\beta}_t\Big)^2 &= \mathbb{E}\Big(\alpha\tilde{X}_t+\delta^2\tfrac{(Z_t-(Y_t+\alpha(T-t)\tilde{X}_t))}{\Sigma_t}\Big)^2\\
&= \mathbb{E}\Big(\tfrac{\delta^2(Z_t-(Y_t+\alpha(T-t)\tilde{X}_t))+\alpha\Sigma_t\tilde{X}_t}{\Sigma_t}\Big)^2\\
&= \Sigma_t^{-2}\mathbb{E}\Big(\delta^2\Big(Z_t-\Big(Y_t+\alpha(T-t)\tilde{X}_t\Big)\Big)+\alpha\Sigma_t\tilde{X}_t\Big)^2\\
&= \Sigma_t^{-2}\mathbb{E}\Big(\delta^2(Z_t-Y_t)+\Big(\alpha\Sigma_t-\alpha\delta^2(T-t)\Big)\tilde{X}_t\Big)^2\\
&= \Sigma_t^{-2}\Big(\delta^4\mathbb{E}\Big((Z_t-Y_t)^2\Big)+\Big(\alpha\Sigma_t-\alpha\delta^2(T-t)\Big)^2\mathbb{E}\Big(\tilde{X}_t^2\Big)\Big)\\
&\quad+\Sigma_t^{-2}\Big(2\delta^2\Big(\alpha\Sigma_t-\alpha\delta^2(T-t)\Big)\mathbb{E}\Big(\tilde{X}_t(Z_t-Y_t)\Big)\Big)
\end{aligned} \tag{27}$$

From the relationship $P_t=\mathbb{E}(X_t^2|\tilde{\mathcal{Y}}_t)-\tilde{X}_t^2$, we have

$$\begin{aligned}
\mathbb{E}\Big(\tilde{X}_t^2\Big) &= \mathbb{E}\Big(\mathbb{E}\Big(X_t^2\Big|\tilde{\mathcal{Y}}_t\Big)-P_t\Big)=\mathbb{E}\Big(\mathbb{E}\Big(X_t^2\Big|\tilde{\mathcal{Y}}_t\Big)\Big)-P_t\\
&= \mathbb{E}\Big(X_t^2\Big)-P_t=\mu_t^2+\varsigma_t-P_t\le\mu_t^2+\varsigma_t<\infty
\end{aligned} \tag{28}$$

where μ_t and ς_t are respectively the unconditional mean and variance of X_t and $\mathbb{E}(P_t)=P_t$ since P_t is deterministic. Next we have

$$\mathbb{E}\Big((Z_t-Y_t)^2\Big)=\mathbb{E}\Big(Y_T-Y_t+\varepsilon M_{(T-t)^\theta}\Big)^2=\mathbb{E}\Big((Y_T-Y_t)^2+\varepsilon^2M^2_{(T-t)^\theta}\Big)<\infty \tag{29}$$

by the Gaussian nature of the variables. Lastly we get

$$\begin{aligned}
\mathbb{E}\Big(\tilde{X}_t(Z_t-Y_t)\Big) &= \mathbb{E}(\mathbb{E}(X_t|\tilde{\mathcal{Y}}_t)(Z_t-Y_t))=\mathbb{E}(\mathbb{E}(X_t(Z_t-Y_t)|\tilde{\mathcal{Y}}_t))\\
&= \mathbb{E}(X_t(\varepsilon M_{(T-t)^\theta}+Y_T-Y_t))=\mathbb{E}(X_t(Y_T-Y_t))=\mathbb{E}(\mathbb{E}(X_t(Y_T-Y_t))|\mathcal{F}_t))\\
&= \mathbb{E}\Big(\alpha(T-t)X_t^2\Big)=\alpha(T-t)\Big(\mu_t^2+\varsigma_t\Big)<\infty
\end{aligned} \tag{30}$$

From 28,29 and 30 we conclude that $\mathbb{E}(\alpha\tilde{X}_t+\delta\tilde{\beta}_t)^2<\infty$. □

Proposition 4.2. *The insider's optimal portfolio strategy is*

$$\pi_t^*=\frac{\alpha\tilde{X}_t+\delta\tilde{\beta}_t}{\delta^2} \tag{31}$$

Proof. The insider's access to privileged information leads to an extra drift term in the dynamic of her risky asset since $dB_t=d\tilde{B}_t+\beta_t dt$. Given the dynamics of her wealth process V_t, it follows that

$$\log(V_T)=\log(v_0)+\int_0^T(\alpha\pi_tX_t+\delta\pi_t\beta_t-\frac{1}{2}\delta^2\pi_t^2)dt+\int_0^T\delta\pi_t d\tilde{B}_t \tag{32}$$

For $\pi\in\mathcal{A}^{\tilde{\mathcal{Y}}}$, we have $\mathbb{E}\Big(\int_0^T\pi_t^2\delta^2dt\Big)<\infty$. The stochastic integral is thus a martingale and this yields $\mathbb{E}\Big(\int_0^T\delta\pi_t d\tilde{B}_t\Big)=0$. Using the Fubini theorem and tower property of conditional expectation and noting that the control π_t is measurable w.r.t the enlarged observation filtration $\tilde{\mathcal{Y}}_t$,we get

$$\begin{aligned}
\mathbb{E}(\log(V_T)) &= \log(v_0)+\mathbb{E}\Big(\int_0^T(\alpha\pi_tX_t+\delta\pi_t\beta_t-\tfrac{1}{2}\delta^2\pi_t^2)dt\Big)\\
&= \log(v_0)+\int_0^T\mathbb{E}\Big[\alpha\pi_tX_t+\delta\pi_t\beta_t-\tfrac{1}{2}\delta^2\pi_t^2\Big]dt\\
&= \log(v_0)+\int_0^T\mathbb{E}\Big[\mathbb{E}\Big(\alpha\pi_tX_t+\delta\pi_t\beta_t-\tfrac{1}{2}\delta^2\pi_t^2\Big|\tilde{\mathcal{Y}}_t\Big)\Big]dt\\
&= \log(v_0)+\int_0^T\mathbb{E}\Big[\alpha\pi_t\mathbb{E}\Big(X_t|\tilde{\mathcal{Y}}_t\Big)+\delta\pi_t\mathbb{E}\Big(\beta_t|\tilde{\mathcal{Y}}_t\Big)-\tfrac{1}{2}\delta^2\pi_t^2\Big]dt\\
&= \log(v_0)+\int_0^T\mathbb{E}\Big[\alpha\pi_t\tilde{X}_t+\delta\pi_t\tilde{\beta}_t-\tfrac{1}{2}\delta^2\pi_t^2\Big]dt
\end{aligned} \tag{33}$$

The portfolio strategy that maximizes $\mathbb{E}(\log(V_T))$ is $\pi_t^*=\frac{\alpha\tilde{X}_t+\delta\tilde{\beta}_t}{\delta^2}$ since π_t^* maximizes the integrand point-wise for all $t\in[0,T]$. From Lemma 4.1, we have that $\mathbb{E}\Big(\alpha\tilde{X}_t+\delta\tilde{\beta}_t\Big)^2<\infty$ and the strategy π_t^* is indeed admissible. □

Corollary 4.3. *The optimal strategy for the regular trader is*

$$\hat{\pi}_t^*=\frac{\alpha\hat{X}_t}{\delta^2} \tag{34}$$

Table 1
Model parameter values.

Volatility δ	1,0.5	Drift α	3
Return's volatility σ	0.5	Noise parameter ε	[0,10]

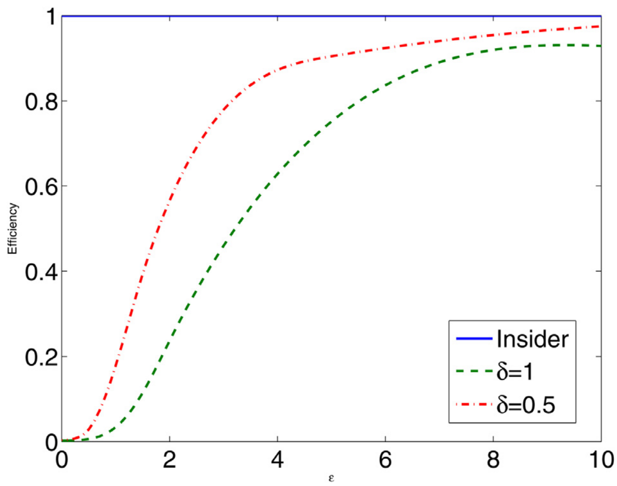

Fig. 1. Efficiency as a function of Noise parameter ε.

Proof. From the dynamics of the regular trader's wealth process, we get

$$\log(V_T) = \log(v_0) + \int_0^T (\alpha\pi_t X_t - \frac{1}{2}\delta^2\pi_t^2)dt + \int_0^T \delta\pi_t dB_t$$

Mimicking the proof of Proposition 4.2 yields

$$\begin{aligned}\mathbb{E}(\log(V_T)) \quad &= \log(v_0) + \mathbb{E}\left(\int_0^T (\alpha\pi_t X_t - \tfrac{1}{2}\delta^2\pi_t^2)dt\right) \\ &= \log(v_0) + \int_0^T \mathbb{E}\left[\alpha\pi_t\hat{X}_t - \tfrac{1}{2}\delta^2\pi_t^2\right]dt\end{aligned} \tag{35}$$

The regular trader's optimal strategy is thus $\hat{\pi}_t^* = \frac{\alpha\hat{X}_t}{\delta^2}$. Similar to lemma 4.1, it is easy to show that $\mathbb{E}\left(\alpha\hat{X}_t\right)^2 < \infty$ and hence the regular trader's optimal strategy is admissible. □

In the following subsection, using numerical computations, we compare the optimal value of the terminal wealth for the insider and the regular trader.

Insider's additional utility

In order to quantify the value of the insider's additional information, we compare the optimal expected utilities for the regular trader and an insider. Let v^I, v^S be the initial capital for the insider and the regular trader respectively. The extra initial capital which the regular trader needs to have in order to obtain the same expected terminal wealth as the insider is given by the difference $v^S - v^I \geq 0$. This difference can be considered as information gain on the part of the insider. Define the ratio

$$\rho = \frac{v^I}{v^S} = \exp\left[-\frac{1}{2\delta^2}\int_0^T \left(\mathbb{E}\left(\alpha\tilde{X}_t + \delta\tilde{\beta}_t^2\right)^2 - \mathbb{E}\left(\alpha\hat{X}_t\right)^2\right)dt\right] \tag{36}$$

as the measure of efficiency for the insider. The measure of efficiency captures the fraction of initial capital that the insider would need compared to the regular trader's initial capital inorder for the two to have equal optimal wealth levels at time T. It can be shown that the integrand is non-negative as it represents the insider's additional utility.

We illustrate numerically the variation of the measure of efficiency at different parameter values. The model parameter values used in the numerical computations are found in Table 1.

Fig. 1 is a plot of efficiency as a function of the noise parameter ε. The solid curve indicates a case whereby $\rho = \frac{v^I}{v^I} = 1$, hence the straight horizontal line. The remaining curves capture variation of efficiency at different values of ε when the stock price volatility is fixed at $\delta = 1$ and $\delta = 0.5$ respectively. As observed, at different fixed values of ε, ρ is higher for $\delta = 0.5$ than when $\delta = 1$. This points to the fact that at higher levels of stock price volatility, the value of the additional information increases and the insider would need a smaller proportion of initial capital compared to the regular trader.

We also note that in both cases when $\delta = 1$ and $\delta = 0.5$, higher values of ε correspond to higher values of ρ, meaning that when the quality of inside information deteriorates, the insider would need a larger proportion of initial capital to match the regular trader's terminal wealth. On the other hand, in the extreme case when $\varepsilon = 0$, both for $\delta = 1$ and $\delta = 0.5$, we have $\rho = 0$. This corresponds to the case when the insider has access to perfect information on the future value of the stock price and there will be existence of arbitrage opportunity, as established e.g. in [1].

Conclusion

In this work, we considered a financial market model whereby the drift of the stock price is unobserved by an insider. Additionally, the insider is able to get privileged information relating to the future value of the stock price. The quality of the insider's privileged information gets better as we move towards the information reveal time. Dynamic enlargement of filtration techniques were applied to incorporate the insider's privileged information. Since the state-space model remains linear in the insider's enlarged observation filtration, Kalman-Bucy filtering techniques were applied inorder to obtain estimates of the unobserved drift process. This work thus fused dynamic enlargement of filtration and stochastic filtering. Subsequently, we obtained explicit analytic results of the optimal portfolio strategy for an insider having the log utility function.

Our numerical results revealed that as the quality of the insider's privileged information deteriorates, the insider would need larger amounts of initial capital (and the regular trader would require less amounts of initial capital) to match the terminal wealth. On the other hand, in the extreme case when the insider receives perfect information on the future value of the stock price, an arbitrage opportunity will exist. Further, the numerical results also reveal the role played by the stock price volatility with regards to the asymmetry of information between the insider and the regular trader, with higher levels of volatility pointing to increased value of the insider's privileged information. Future research may consider other utility functions e.g the power and exponential utilities.

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CRediT authorship contribution statement

Stanley Sewe: Conceptualization, Writing – original draft, Writing – review & editing. **Philip Ngare:** Methodology, Supervision, Writing – review & editing. **Patrick Weke:** Methodology, Supervision, Writing – review & editing.

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